In presenting the dissertation as a partial fulfillment of the requirements for an advanced degree from the Georgia Institute of Technology, I agree that the Library of the Institute shall make it available for inspection and circulation in accordance with its regulations governing materials of this type. I agree that permission to copy from, or to publish from, this dissertation may be granted by the professor under whose direction it was written, or, in his absence, by the Dean of the Graduate Division when such copying or publication is solely for scholarly purposes and does not involve potential financial gain. It is understood that any copying from, or publication of, this dissertation which involves potential financial gain will not be allowed without written permission.

3/17/65
b

SYNTHESIS OF A CLASS OF HYBRID TRACKING SYSTEMS

A THESIS

Presented to

The Faculty of the Graduate Division

by

Jon Willard Petway

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy in the

School of Electrical Engineering

Georgia Institute of Technology

May, 1967

SYNTHESIS OF A CLASS OF HYBRID TRACKING SYSTEMS

Approved:

Date approved by Chairman: 5-3-67

ACKNOWLEDGMENTS

To use a trite phrase accurately, it is indeed a pleasure to express appreciation to my thesis advisor, Dr. R. P. Webb, for his guidance during the development of this work. Without his attention to detail it probably would not have been successful. I also thank Dr. J. B. Peatman and Dr. T. M. White, who served on the reading committee.

I sincerely appreciate the help of Dr. B. J. Dasher and Dr. J. L. Hammond, who made possible the financial support of this thesis. Dr. Hammond also helped in the formulation of the problem on which it is based.

Finally, I extend extreme gratitude to my family, who suffered considerable neglect during my entire tenure as a graduate student, especially during the thesis research and writing period.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

SUMMARY

This thesis reports the development of a procedure for the synthesis of a class of hybrid tracking systems. The class of systems considered contain a fixed and unalterable element, referred to as the plant, whose output variable is to track an available reference signal. The part of the system for which a synthesis procedure is developed is that part of the system which generates a driving signal for the plant. An algorithm is determined for generating the plant driving signal from reference signal and plant output quantities. Specification of this algorithm completes the system design. It is assumed that the implementation of this algorithm will be digital. The plant is assumed to be a continuous device. The system, containing both continuous and discrete interconnected subsystems, is therefore termed a hybrid tracking system.

The synthesis procedure presented here is based on a new combination of known techniques. Advantage is taken of both the newer time domain techniques and the older, but equally powerful for some purposes, frequency domain techniques. Modern synthesis methods can be applied only if the reference signal is a member of a certain limited class of signals. The innovation on which this thesis is based consists of determining a system for several reference signals which are members of this class and then seeking a system to be used with the sum of these reference signals, a reference signal which, in general, is not a member of the above mentioned class. The set of reference signals for which systems

are synthesized consists of sinusoids, and this permits utilization of some frequency domain techniques.

Tracking system synthesis consists of finding the algorithm by which the plant driving signal should be generated from reference signal and plant output quantities. This determination is formulated as a double minimization problem. It is assumed that the reference signal can be approximated by a finite number of sinusoidal terms. The amplitudes and frequencies of these terms are determined from knowledge of the power spectrum of the reference signal. For the case of a sinusoidal reference signal, an optimum tracking system can be determined using functional minimization techniques. The optimum system is the one which results in a minimum value for a preselected functional of tracking error. This functional is referred to as the index of system performance. An optimum tracking system is determined for each frequency present in the approximation of the reference signal, thus determining optimum system structure as a function of reference signal frequency. These optimum systems are feedback systems, and it is shown that the feedback structure and parameters are independent of reference signal frequency. Only the input gains of the system (coefficients of the reference signal in the algorithm for computing the plant driving signal) change with reference signal frequency. A second minimization problem is formulated for use in selecting the best set of fixed input gains for use with the actual reference signal (containing an entire band of frequencies). The input gains are selected in such a way that a pre-selected measure of deviation of the actual system from the optimum over the frequency band containing the reference signal is minimized. In the definition of this measure, deviations at various frequencies are weighted according to the relative amount of energy contained in the reference signal in the neighborhood of the frequency in question. An alternate approach to this second minimization is presented. This approach is based on the criterion for distortionless transmission.

The following outline summarizes the steps which must be completed in an application of the synthesis procedure developed in this thesis.

1. Determination of the frequencies and amplitudes of the terms of the approximating sum for the reference signal.
2. Determination of the optimum system for tracking each frequency in the approximating sum. This step determines the feedback structure and parameters of the tracking system.
3. Determination of fixed input gains to be used for tracking the actual reference signal. This determination is based on the optimum input gain versus reference signal frequency information obtained in step 2 above.

The main advantage of this method is that it permits tracking system design to be accomplished as a synthesis rather than a repeated analysis procedure without severe restriction on the class of reference signals for which the procedure is applicable. The class of problems toward which the method is aimed, namely hybrid system design, has not been investigated in the detail demanded by the rate of utilization of such systems. The method serves a useful purpose also in that it shows how to make practical use of functional minimization techniques in the solution of an important engineering problem, and thereby illustrates a method of attack which could be applied to other problems. The method is limited to application in cases where the power spectrum of the reference signal is known. This is not a severe limitation because in many cases the power spectrum of the reference signal can be estimated. The method is also restricted in that it is assumed that the plant can be modeled by a set of linear ordinary differential equations.

Two sensitivity analyses are necessary because of the digital nature of the device which performs the computation indicated by the algorithm. System performance sensitivity to quantization and changes in sampling period length is investigated. To judge performance sensitivity to quantization two quantities are determined. These are a conservative bound on tracking error due to quantization and a statistical description of tracking error due to quantization. A measure of how system performance is affected by changes in sampling period length is obtained by investigating the changes in frequency response of the system as the sampling period length is varied. An expression for the frequency response, as it depends on sampling period length, is written. This expression is plotted in sample problems.

For use in the alternate approach to the second minimization and in the sensitivity analysis, a technique is developed which is useful in many situations. This technique is utilized in only a limited way in this work, but would have a wide range of possible applications. In the alternate approach as well as the sensitivity analysis there is need to compute the frequency response of the tracking system. The system model is a vector difference equation. A method is developed for determination of system frequency response from this model. Such a method has not been found in the literature. It furnishes new insight into the relationship between modern state variable and classical frequency domain descriptions of a linear system.

Sample problems are posed and solved to illustrate the steps of the synthesis procedure.

CHAPTER I

INTRODUCTION

The Problem

Many situations arise wherein it is necessary to cause the output of some known dynamic system to track an available reference signal. In general the reference signal is time variable and not known in advance. The dynamic system may be dictated by the application. For example, in a radar system the antenna must be chosen from consideration of its transmitting and receiving capabilities. Tracking system design then consists of specifying auxiliary equipment to be used in conjunction with the fixed dynamic system, referred to as the plant, to cause the system output to track the reference signal. The synthesis method developed in the following pages is applicable in such situations.

The plant is assumed to be a device whose output is related to its input in a known manner. There may be multiple inputs. In the radar example the output would be antenna position and there would be three inputs corresponding to the voltages applied to the motors which rotate the antenna about its three axes. If the plant output is to execute certain motion this must be brought about by selection of the plant input. The plant input will be called the control signal since it is the only means of controlling the plant output. The auxiliary equipment which generates the control signal is called the control element of the tracking system.

Tracking system synthesis consists of determination of the structure and parameters of the control element. System synthesis is formulated as a double minimization problem, and solution of this problem shows that for minimum mean square tracking error the control signal should be a certain linear combination of plant and reference signal data. Thus, the control element is specified by an algorithm for computing the control signal. It is assumed that this computation will be implemented digitally. Hence the term hybrid tracking system. Sensitivity analyses dictated by the digital nature of the control element are included.

The motivation for the present work comes mainly from two facts. The first of these is that it is now becoming practical to use digital equipment in applications such as hybrid tracking systems. As the technology of digital equipment advances, more economical equipment, such as the special purpose computer which is specified in this synthesis method, is becoming available. This brings forth new applications and therefore spurs the development of the technology even further. An equilibrium should not be expected in the near future. Development of mathematical methods for analysis and synthesis of hybrid systems is lagging. The second motivating fact for this work is that it can be generalized to form the basis of a synthesis method for continuous (non-hybrid) tracking systems. Synthesis of continuous tracking systems has been given considerable attention in the literature. However, the method of attack used in this work is somewhat novel and could provide additional insight into the synthesis of continuous tracking systems.

History of the Problem

One of the earliest treatments of sampled-data systems, a special case of hybrid tracking systems, is found in a textbook by Oldenbourg and Sartoris.[1] These authors analyzed such systems by developing and then solving a difference equation model. The model which was developed is similar, in some respects, to the modern state variable models of such systems.

A more popular model for sampled-data systems was subsequently introduced by Barker.[2] His model, based on the Z-transform, was an extension of well developed models, based on the Laplace transform, for continuous (analog) systems. Control element design for continuous tracking systems at that time consisted of assuming a form for the control element and then performing a stability analysis and an analysis of system performance with certain test reference signals, such as the unit step.[3] If the system performed satisfactorily in the judgment of the designer then it was used. If not, modifications based on the experience and intuition of the designer were introduced and the analysis repeated. In this way a useful system could usually be determined. Extensive techniques were available for the analyses mentioned. Most of these techniques were modified to permit the analysis of sampled-data systems modeled in terms of the Z-transform.[4] However, these modified techniques and the Z-transform model are complicated, difficult to apply, and do not provide the physical insight present in the case of continuous systems. It is obvious that assuming the form of the control element as a starting point in its design is a severe restriction on the design process. Classical design methods are extremely difficult to apply to hybrid systems because of the complicated mathematical models and methods which must be used.

A more modern approach to tracking system design would leave the form of the control element to be determined completely by the design routine. The problem of control element design can then be considered from a synthesis, rather than a design by repeated analysis, point of view. The resulting tracking systems have, in general, a substantially different character and configuration and give better performance than those determined by the trial and error process. In this more recent approach a mathematical statement is made of what an ideal system would accomplish. The mathematical statement consists of defining an index of system performance which is a functional of the tracking error. System synthesis is then formulated as a variational problem based on minimizing this performance index. This approach to system design is the innovation of Weiner[5] who first used the technique in the synthesis of the well known optimum linear filter. Bellman[6] has developed a method of solving variational problems such as are encountered in tracking system synthesis. State variable representation of the system is amenable to the application of this method. The solution of the variational problem determines the plant driving signal which results in a minimum value for the performance index. For a general and useful class of performance indices this control signal will be a linear combination of the plant state vector and the reference signal.[7] Hence the control element, and therefore the system, is specified by the algorithm of this linear combination.

The original development of modern synthesis techniques was for continuous systems. Some of these techniques have been extended to discrete systems with no loss in physical insight and no complication as was present in the extension of classical frequency domain techniques. Kalman and Bertram[8] have published a detailed investigation of state modeling as applied to hybrid systems. Tou[9] has applied Bellman's dynamic programming method to the synthesis of hybrid tracking systems. Dynamic programming is particularly suitable for application to hybrid system synthesis. Tou's result is severely restricted, however, in that he considered only systems with non-time variable reference signals. Such systems are referred to as regulator systems and are not really tracking systems in the strictest sense.

Application of functional minimization techniques to tracking system synthesis requires some information concerning the behavior of the reference signal. This information is needed in order to write a performance index. Kalman and Koepke[10] have investigated some aspects of the formulation of performance indices for use in tracking system synthesis. Their work shows that a physically meaningful and mathematically feasible performance index can easily be written for use in regulator synthesis. In this same paper it is also shown that if the reference signal is the solution of a known differential equation then a performance index having the same form as in the case of a regulator can be written. It can be said that, in general, tracking system synthesis by application of functional minimization techniques requires analytic knowledge of the reference signal.

A technique due to Kalman[11] permits tracking system synthesis when statistical knowledge of the reference signal is available. This technique can be used with both deterministic and non-deterministic reference signals. The technique consists of finding a linear filter, or differential equation, which when excited by white noise produces an output having the known statistics of the reference signal. With knowledge of this differential equation it is possible to write down a performance index which can be used to find a tracking system which produces, on the average, minimum tracking error. Methods exist for finding the required differential equation from the reference signal statistics. Although Kalman originally applied this technique to synthesis of systems whose purpose was separation of signal from signal plus noise, it has since been applied to tracking system synthesis.[12]

While the synthesis of tracking systems has been investigated in some detail, the techniques which have been developed are in some cases very restrictive as in the case of regulator design where it is assumed that the reference signal will not be time variable. The more general techniques, such as the one due to Kalman previously mentioned, sometimes have serious disadvantages. For example, finding the required differential equation for application of this technique may be a considerable task. Most of the tracking system synthesis techniques developed to present have been specifically applicable to continuous systems. Most of these techniques can easily be modified to permit synthesis of hybrid tracking systems. Sometimes, however, difficulties are encountered in this extension.

The method developed in the following pages is a new approach to tracking system synthesis which is specifically applicable to the synthesis of hybrid tracking systems. It is based on the fact that if the reference signal solves a known differential equation, then an optimum tracking system can be determined. Using this fact the optimum system can be studied as a function of reference signal frequency. It is shown that the feedback structure of the optimum system does not change with reference signal frequency. Optimum input gains do, however, vary with frequency. A second minimization is used to select fixed input gains for use with a band of frequencies. The method cannot be thought of as an extension of existing methods even though known techniques are exploited. It is a new and different approach to the synthesis problem. Including a sensitivity analysis, the method is more complete than some existing methods. The amount of attention presently being devoted to sensitivity analysis in the literature shows that it is being realized that systems designed by application of optimization theory may be of no practical use because of sensitivity problems. A sensitivity analysis serves to indicate such problems in the design stage where they may be dealt with.

CHAPTER II

TRACKING SYSTEM SYNTHESIS AS A DOUBLE MINIMIZATION PROBLEM

The purpose of this chapter is to present a more detailed description of the problem and to show how it can be formulated as a double minimization problem. For this purpose it is necessary to discuss the system structure in some detail. Also the problem of reference signal modeling is discussed.

A hybrid tracking system is shown in block diagram form in Figure 1. Two elements, one continuous and one discrete, are interconnected and this interconnection constitutes a hybrid system. The double lines in Figure 1 are intended to indicate the fact that the variables are vectors (state variables). This usage is becoming fairly standard. The plant can be considered as a constraint on system design. That is, the plant must be accommodated by the synthesis procedure. The plant is a continuous device and could be simulated on an analog computer. The adjective analog is attached to the plant, and it is referred to as the analog part of the system. The control element has the function of generating the control signal. This is accomplished by combining available data according to an algorithm. It is assumed that this computation will be implemented by digital means and that the control signal will remain constant while a new value for it is being computed. Therefore the present value of this signal may be used in the computation of the

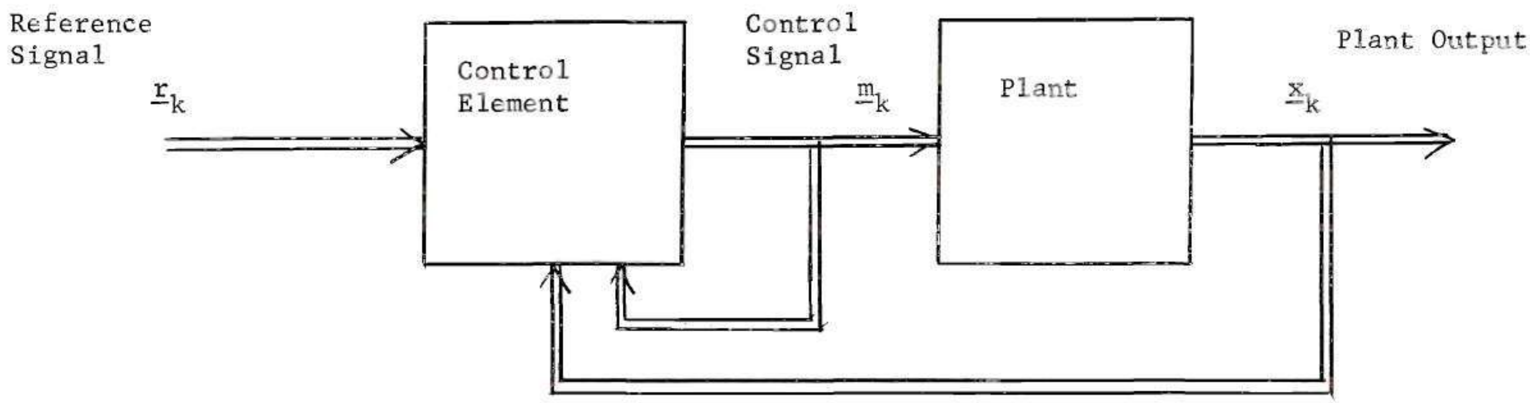


Figure 1. Block Diagram of Hybrid Tracking System.

next value. The time required for computation of the control signal is called the sampling period length and it is symbolized by T. This is the interval between changes in the control signal. Since the system contains both analog and digital subsystems and since the purpose of the system is tracking, the term hybrid tracking system is appropriate.

In Figure 1 all signals are represented as vectors. Representation of the control signal as a vector simply accounts for the fact that the plant may have more than one input. If it has only one input then the control signal vector has only one component. The control signal is represented as a sequence. This is because of its discrete nature. The symbol m_k represents the value of this signal between the $k^{\underline{th}}$ and $(k+1)^{\underline{st}}$ sampling instants. The vector representing the plant output is the plant state vector. Its components can be thought of as the integrator outputs of an analog simulation of the plant. The reference signal vector is assumed to be two dimensional consisting of the reference signal and its time derivative. The system is considered to be tracking perfectly when it maintains the first two components of the plant state vector equal to the two components of the reference signal vector.

The control element, the digital portion of the tracking system, is the system element for which a synthesis procedure has been developed. The purpose of this element is to produce a driving signal, the control signal in Figure 1, for the plant. This signal is the only available means of influencing the plant output and must be carefully determined if the plant output is to track the reference signal. The synthesis procedure results in specification of an algorithm. At integer multiples of the sampling period the system variables are sampled. These sampled values are combined according to the algorithm to produce the control signal.

Unless the reference signal over the entire life of the tracking system is known at the time the system is being designed it is impossible to determine a control element which would lead to perfect tracking. Even if the reference signal were known beforehand, an extremely sensitive time variable control element would be called for to produce perfect tracking. In general, the reference signal will not be known a priori and a time variable control element will not be acceptable. Therefore some model of the reference signal is required for design purposes which leads to a fixed control element which will cause the system to track reasonably well, though not perfectly, a useful class of reference signals.

The modern approach to tracking system design consists of defining an index of system performance which is an error functional and using variational techniques to find the control signal which causes this performance index to assume a minimum value. The most common performance index is mean square error. This quantity, in general, is minimized by a control signal which can be generated by a linear feedback structure. Definition of a performance index which is meaningful and at the same time leads to a workable variational problem is probably the most troublesome step encountered in the application of modern design techniques. Such a definition hinges on the expression or model of the reference signal. The reference signal expression must be such that it can be used in the definition of a useful performance index. A model of the reference signal means any expression which can be used to describe the essential features of the reference signal.

Many reference signal models have been considered by various contributors. Some of these have been mentioned in Chapter I. They include constant or non-time varying signals, where system performance is judged by the step response, and the statistical models where the reference signal is represented as the output of a linear filter excited by white noise (an initial condition). In classical control theory much attention is given to system frequency response. System design can be based on shaping the frequency response according to such criteria as degree of stability, distortion, steady state error, noise rejection, etc. or a combination of these. If a system has unity gain at all frequencies contained in the system input and phase shift which is proportional to frequency , constant time delay , then the system output is a distortionless reproduction of its input.

If the reference signal is a solution of an unforced differential equation then a mean square error performance index can be written and variational methods applied to find the control signal for minimum mean square error. The minimizing control signal can be realized as a linear combination of plant and reference signal variables. That is, the optimum system is a linear feedback system.

Since a sinusoid is a solution of a differential equation, an optimum tracking system can be determined when the reference signal is a sinusoid. The optimum system will depend on the frequency of the sinusoidal reference signal. Now if the actual reference signal contains several frequencies, an optimum system can be determined for each of these frequencies and a study can be made to investigate whether a single system structure exists which could be used with the actual reference signal. In fact, if the reference signal contains a continuous spectrum of frequencies an optimum system can be determined for several frequencies in the spectrum and a single system can be sought which performs satisfactorily over the entire spectrum. Such studies are the basis of the synthesis method being presented here.

Determination of system structure for minimum mean square error with a sinusoidal reference signal has been formulated as a variational problem and this variational problem has been solved. The solution of this problem requires that a set of simultaneous non-linear algebraic equations be solved. Numerical techniques have been developed for solving this set of equations. It has been shown that the feedback structure of the optimum system does not change with changes in reference signal frequency. This fact is very important because it allows practical implementation of the synthesis procedure. If the feedback gains of the optimum system varied with reference signal frequency the problem of selecting a set of gains for use with a reference signal containing several frequencies would be untractable.

Thus a first minimization, the minimization of mean square error, results in a fixed feedback structure not depending on reference signal frequency. However, optimum system input gains, of which there are two, one for each component of the reference signal vector, do change with reference signal frequency. A method is needed for selecting the best set of input gains for use with the actual reference signal which contains many frequencies. This selection is formulated as a second minimization problem. The input gains are chosen so that the system most closely approximates the optimum structure at a selected set of frequencies in the band of the actual reference signal. An error index is defined which depends on the deviation of the system from the ideal at each of these frequencies. Both amplitude and phase deviations enter into this error index. System input gains are selected which minimize the error index. Selection of the input gains completes the design procedure.

An alternate method of selecting input gains is presented. This alternate method is based on minimization of an error index which depends on the deviation of the system from an ideal system having unity gain and constant time delay at all frequencies in the band containing the reference signal.

CHAPTER III

MATHEMATICAL FORMULATION OF THE PROBLEM

It has been pointed out in the last chapter that synthesis of hybrid tracking systems can be formulated as a double minimization problem. The first of these minimization problems will be solved using modern mathematical methods which necessitate a certain type model for the plant and control element and require certain definitions. In this chapter the necessary models and definitions will be discussed, and the synthesis problem will be formulated such that variational techniques can be applied to find an optimum tracking system.

Mathematical Modeling

In order to develop a model for the system which is amenable to the application of techniques which will be used to effect system synthesis it is necessary to use some sort of model for the control element. The quantity determined by the synthesis procedure is actually the plant input or control signal. The control element is specified as a means of realizing the control signal. It will be assumed here that the control signal will be a linear combination of system variables and therefore that the control element is described by a linear difference equation. This model of the control element is subject to verification later.

Since the control element is of a discrete nature it must be modeled by a difference equation or recursion relationship. The plant is of a continuous nature, and it is assumed that a differential equation model of this system element is known. These two system elements are interconnected as shown in Figure 1. With one modeled by a difference equation and the other modeled by a differential equation, there is no compact way of imposing the interconnection constraint. Both must be modeled by the same type equation for this purpose. It is not possible to convert the control element difference equation to a differential equation without making questionable assumptions. However, because the control signal does not change between sampling instants, it is possible to obtain a recursion relationship which involves the plant states at sampling instants and the sequence of control signals. This relationship constitutes a difference equation model of the plant and makes possible a mathematical expression of the interconnection of the two system elements.

In view of this it is possible to write a difference equation model for the overall system. This model expresses the interrelationship of all system variables and their relationship to the reference signal. It is a model suitable for use in functional minimization techniques which will be exploited in the development of the synthesis procedure.

If the reference signal is a sinusoid, which satisfies an undamped second order differential equation, then it is possible to write a vector difference equation which gives the values of the reference signal and its time derivative at any sampling instant in terms of their values at the previous sampling instant. Reference signal frequency and sampling period length are parameters in this equation. Such an equation permits modeling of the entire tracking system, including the reference signal, by a homogeneous difference equation. This fact is extremely important in writing a feasible performance index as will be shown in the next section.

The Plant

This system element is the tracking vehicle and is selected before the synthesis procedure is applied. A dynamic model of the plant must be available for tracking system design purposes. This dynamic model, a set of linear ordinary differential equations, will, in any application, be used to obtain a difference equation model. The general procedure to be followed is outlined below.

The plant could consist of a single device having one input and one output. It could consist of several interconnected devices each having a single input and a single output. Only one output would be considered a system output. That is only one variable tracks the reference signal. It is not possible here to consider the full generality of conversion of the plant model to a discrete model. Two special cases will be sufficient to indicate the necessary steps.

Case I. In this case the plant has one input and one output. These two variables are related by a differential equation of the form

$$\frac{d^n X}{dt^n} + a_{n-1} \frac{d^{n-1} X}{dt^{n-1}} + \ldots + a_o X = K\, m(t) \qquad (1)$$

Well known techniques[12] can be used to convert this $n^{\underline{th}}$ order equation to a set of n simultaneous first order equations. For example let the following definitions be made.

$$X = X_1$$

$$\dot{X}_1 = X_2$$

$$\vdots$$

$$\dot{X}_{n-1} = X_n$$

Then it is true that

$$\dot{X}_n = -a_{n-1}X_n - \dots - a_1X_2 - a_oX_1 + K\,m(t)\,.$$

This is a set of n simultaneous first order equations which can be written in vector-matrix form. Thus

$$\underline{\dot{X}} = A\,\underline{X} + \underline{b}\,m \tag{2}$$

where

$$\underline{X} = \begin{bmatrix} X_1 \\ X_2 \\ \cdot \\ \cdot \\ \cdot \\ X_n \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ & & & \ddots & 1 \\ -a_o & -a_1 & & & -a_{n-1} \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ K \end{bmatrix}.$$

The solution of (2) can be shown[12] to be given by the following vector $\underline{X}(t)$.

$$\underline{X}(t) = e^{At}\,\underline{X}(t_o) + \int_{t_o}^{t} e^{A(t-s)}\underline{b}\,m(s)\,ds \tag{3}$$

In this equation t_o is the initial time and $\underline{X}(t_o)$ is the vector initial condition. Also

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \dots + \frac{A^nt^n}{n!} + \dots$$

The fact that the output of the digital control element changes only at discrete instants makes it possible to convert (3) to a difference equation. Let

$$m(t) = m_k \qquad kT \leq t < (k+1)T$$

$$\underline{X}(kT) = \underline{X}_k$$

$$t_o = kT \ .$$

Then (3) becomes

$$\underline{X}(t) = e^{A(t-kT)} \underline{X}_k + \int_{kT}^{t} e^{A(t-s)} \underline{b} \, ds \, m_k$$

The change of variable $t - s = \alpha$ leads to

$$\underline{X}(t) = e^{A(t-kT)} \underline{X}_k + \int_0^{t-kT} e^{A\alpha} \underline{b} \, d\alpha \, m_k$$

which may be evaluated at $t = (k+1)T$.

$$\underline{X}_{k+1} = e^{AT} \underline{X}_k + \int_0^T e^{A\alpha} \underline{b} \, d\alpha \, m_k \tag{4}$$

Equation (4) constitutes a difference equation model of the plant dynamics. By letting

$$e^{AT} = \varphi_p, \quad \int_0^T e^{A\alpha} \underline{b} \, d\alpha = \underline{\Delta}_p$$

a more compact expression can b e written;

$$\underline{X}_{k+1} = \varphi_p \underline{X}_k + \underline{\Delta}_p \, m_k \tag{5}$$

<u>Case II</u>. In this case a plant with two inputs is considered. The plant contains two separate devices each with one input and one output. These devices are connected together as shown in Figure 2b.

This configuration is chosen for illustration because of the large number of other useful configurations which can be generated from it.

The functions $m_1(t)$ and $m_2(t)$ are the inputs and it is assumed that these are constant except at sampling instants. Thus

$$\left.\begin{array}{l} m_1(t) = m_{1k} \\ m_2(t) = m_{2k} \end{array}\right\} \quad kT \leq t < (k+1)T$$

A vector-matrix differential equation can be written for each device.

$$\dot{\underline{X}} = A_x \underline{X} + \underline{b}_x m_3(t) \tag{6}$$

$$\dot{\underline{Y}} = A_y \underline{Y} + \underline{b}_y m_2(t) \tag{7}$$

The subscript X on A_x indicates the equation of the device having X(t) as its output. The solutions of these equations are

$$\underline{X}(t) = e^{A_x(t-kT)} \underline{X}_k + \int_{kT}^{t} e^{A_x(t-s)} \underline{b}_x m_3(s)\, ds \tag{8}$$

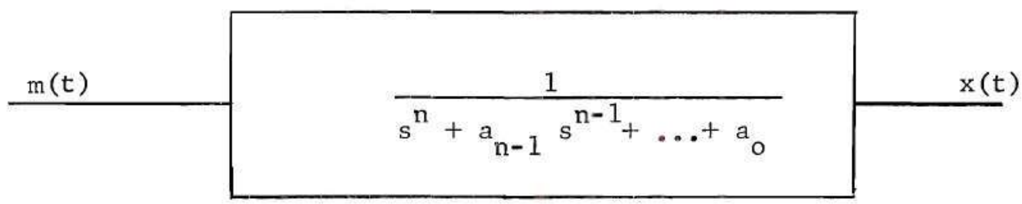
$$\underline{Y}(t) = e^{A_y(t-kT)} \underline{Y}_k + \int_{kT}^{t} e^{A_y(t-s)} \underline{b}_y\, ds\, m_{2k} \tag{9}$$

Equation (9) can be treated as in case I to obtain

$$\underline{Y}_{k+1} = \varphi_y \underline{Y}_k + \Delta_y m_{2k}$$

Since

$$m_3(t) = m_{1k} + y(t)$$



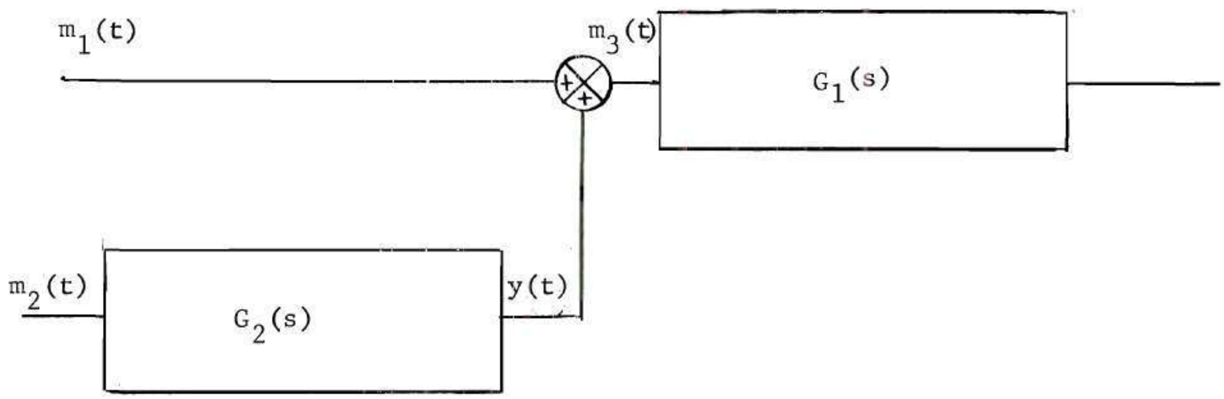


Figure 2. Tracking System Plants for Cases I and II.

equation (8) can be written

$$\underline{X}(t) = e^{A_x(t-kT)}\underline{X}_k + \int_{kT}^{t} e^{A_x(t-s)}\underline{b}_x[m_{1k} + \underline{V}'\underline{y}(s)]ds$$

where

$$V = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

It is assumed that the first component of $\underline{y}(t)$ is $y(t)$. This will be true if the pattern of case I is followed. By routine steps it can be shown that $\underline{X}_{k+1}$ is of the form

$$\underline{X}_{k+1} = \varphi_x\underline{X}_k + \varphi_y\underline{Y}_k + \underline{\Delta}_1 m_{1k} + \underline{\Delta}_2\, m_{2k} \tag{10}$$

where

$$\varphi_x = e^{A_xT}, \; \varphi_y = e^{A_yT}, \;\; \underline{\Delta}_1 = \int_0^T e^{A_x\alpha}\, b_x d\,\alpha \,,$$

The entire plant is then represented by

$$\begin{bmatrix} \underline{X}_{k+1} \\ \underline{Y}_{k+1} \end{bmatrix} = \begin{bmatrix} \varphi_x & \varphi_y \\ 0 & \varphi_y \end{bmatrix} \begin{bmatrix} \underline{X}_k \\ \underline{Y}_k \end{bmatrix} + \begin{bmatrix} \Delta_1 & \underline{\Delta}_2 \\ 0 & \underline{\Delta}_y \end{bmatrix} \begin{bmatrix} m_{1k} \\ m_{2k} \end{bmatrix} \tag{11}$$

which is a set of first order simultaneous difference equations of the same general form as (5) except in these are multiple inputs. That is, the control signal is a vector.

The two cases above illustrate the general procedure for obtaining a difference equation model for the plant. In any application of the synthesis procedure it is necessary that such a model be determined.

The Control Element

It was pointed out at the beginning of this section that the form of the control element model must be stated here and shown to be correct later. The model which will be used is a linear model. This is made to seem somewhat less arbitrary by realizing that, in general, mean square tracking error is minimized by a linear feedback system.[10]

The model of the control element will thus be of the form

$$\underline{m}_{k+1} = \varphi_D \underline{m}_k + \Delta_{D1} \underline{X}_k + \Delta_{D2}\, \underline{r}_k \tag{12}$$

where φ_D is a matrix, Δ_{D1} and Δ_{D2} are row vectors, $\underline{X}_k$ is the plant state vector, $\underline{m}_k$ is the control vector, and $\underline{r}_k$ is a two dimensional vector whose components are the reference signal and its time derivative.

The objective of the synthesis procedure is to determine φ_D, Δ_{D1}, and Δ_{D2}. Determination of these quantities specifies the structure and parameters of the control element.

The Interconnected System

A difference equation model of the entire tracking system interconnected as shown in Figure 1 is easily written from the plant and control element models-equations (11) and (12) respectively. Thus,

$$\begin{bmatrix} \underline{X}_{k+1} \\ \underline{m}_{k+1} \end{bmatrix} = \begin{bmatrix} \varphi & \Delta \\ \Delta_{D1} & \varphi_D \end{bmatrix} \begin{bmatrix} \underline{X}_k \\ \underline{m}_k \end{bmatrix} + \begin{bmatrix} 0 \\ \Delta_{D2} \end{bmatrix} \underline{r}_k \tag{13}$$

represents the entire system.

In the important special case where the reference signal r(t) is a sinusoid it is true that

$$\ddot{r} + \omega^2 r = 0$$

where ω is the frequency of r(t). Letting

$$r = r_1$$

and defining $$\dot{r}_1 = r_2$$

it follows that

$$\dot{r}_2 = -\omega^2 r_1$$

these two equations can be written in matrix form as

$$\begin{bmatrix} \dot{r}_1 \\ \dot{r}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$$

which can be solved as in equation (3) to show that

$$\begin{bmatrix} r_1(t) \\ r_2(t) \end{bmatrix} = \begin{bmatrix} \cos \omega(t - t_o) & \frac{1}{\omega} \sin \omega(t-t_o) \\ -\omega \sin \omega(t - t_o) & \cos \omega(t-t_o) \end{bmatrix} \begin{bmatrix} r_1(t_o) \\ r_2(t_o) \end{bmatrix}$$

Letting $t_o = kT$ and $t = (k+1)T$ leads to

$$\underline{r}_{k+1} = \begin{bmatrix} \cos \omega T & \frac{1}{\omega} \sin \omega T \\ -\omega \sin \omega T & \cos \omega T \end{bmatrix} \underline{r}_k$$

or in shorter notation

$$\underline{r}_{k+1} = \varphi_j \underline{r}_k \tag{14}$$

This recursion relationship for $\underline{r}_k$ makes it possible to model the entire tracking system, including the reference signal, as a homogeneous difference equation.

$$\begin{bmatrix} \underline{X}_{k+1} \\ \underline{r}_{k+1} \\ \underline{m}_{k+1} \end{bmatrix} = \begin{bmatrix} \varphi & 0 & \Delta \\ 0 & \varphi_j & 0 \\ \Delta_{D1} & \Delta_{D2} & \varphi_D \end{bmatrix} \begin{bmatrix} \underline{X}_k \\ \underline{r}_k \\ \underline{m}_k \end{bmatrix} \tag{15}$$

This completes the discussion of mathematical modeling techniques. The techniques which have been discussed are sufficient to permit the development of the synthesis procedure.

An Index of System Performance

The performance index is the quantity which is used to gauge system performance. It serves as a yardstick by which the quality of system performance can be judged, and, more important than this, it provides a means by which different systems can be compared. Such a comparison is the basis of most tracking system synthesis methods. The performance index is defined to be some meaningful quantity which depends on error and mathematical techniques are used to find the particular system which has a minimum performance index and therefore minimum error. This particular system is considered the synthesized system and the methods used for its determination constitute a synthesis procedure.

If the synthesis problem is to be successfully solved the performance index must possess two qualities. First, it must be a meaningful quantity. That is, it must be a quantity a minimum value of which insures good tracking system performance. Second, it must be a quantity which lends itself to the mathematical comparison techniques which are necessary. It must be possible to use the performance index in a mathematical formulation of the synthesis problem.

The purpose of this section is to develop and discuss the performance index which will be used in the first of the two minimizations by which a hybrid tracking system will be determined. In the first minimization the particular system which performs best, as indicated by a minimum value of the performance index, with a sinusoidal reference signal will be determined. The performance index which will be used is a functional of the tracking error. In fact it will be the sum of the squares of the tracking errors at the sampling instants. These errors will be summed from the instant when tracking commences until infinite time has elapsed. A mathematical expression for this sum is

$$S = \sum_{j=o}^{\infty} [(X_{1j} - r_{1j})^2 + \lambda(X_{2j} - r_{2j})^2] \tag{16}$$

In addition to depending on tracking error, this quantity depends on the difference in derivatives of the plant output and the reference signal. The argument in favor of this is that differences in rates of these variables indicates future tracking error. The parameter λ allows arbitrary weighting of this type of error. The performance index S can be written more compactly in terms of vector variables.

$$S = \sum_{j=o}^{\infty} \underline{Z}_j{}' \, Q \, \underline{Z}_j \tag{17}$$

where

$$\underline{Z}_j = \begin{bmatrix} X_{1j} \\ X_{2j} \\ \vdots \\ X_{nj} \\ r_{1j} \\ r_{2j} \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & -1 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & -1 \\ 0 & 0 & 0 & & 0 & 0 & 0 \\ \vdots & \vdots & & \ddots & \vdots & & \vdots \\ & & & & 0 & & \\ -1 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & -1 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}$$

This performance index is well known. It is meaningful because it depends on mean square error, a non-negative monotone increasing functional of tracking error. It also leads to a workable mathematical formulation of the synthesis problem. The sum over all positive values of j, all sampling instants after tracking begins, leads to a system with no time variable gains. In some synthesis procedures it is undertaken to find a system which minimizes plant input energy. A term of the form

$$\sum_{j=o}^{\infty} \underline{m}_j{}' H \, \underline{m}_j$$

where

$$H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is included in the performance index. With an infinite sum for the performance index such a term would cause unboundedness unless the control signal approaches zero. A term of this form will be included for generality, but, in general, H must be a null matrix in the synthesis procedure developed here.

The complete performance index which will be used in the present work is then

$$S = \sum_{j=o}^{\infty} [\underline{Z}_j{}' Q \underline{Z}_j + \underline{m}_j{}' H \underline{m}_j] \tag{18}$$

CHAPTER IV

SYSTEM SYNTHESIS

The performance index defined in equation (18) is a measure of how well the system performs. Perfect tracking at all sampling instants would lead to a value of zero for the performance index. It should not be expected that a system can be found which performs perfectly. Therefore the system will be sought which minimizes tracking error. Once a method has been developed for this determination for a sinusoidal reference signal, optimum tracking system structure can be investigated as a function of reference signal frequency. The optimum system is that system which yields minimum tracking error as measured by the performance index. It will be shown that the optimum system is a linear feedback system and that the parameters of the optimum feedback structure do not vary with the frequency of the reference signal. Selection of the input gains, or parameters, is accomplished by a second minimization. An alternate approach to the problem of selecting input gains is presented in the last section of this chapter.

Dynamic Programming

Equation (18) defines a functional. That is, the performance index is a quantity which depends on a function, or in this case a sequence. Although the performance index is a sum of squares of tracking errors, it in fact depends on the sequence of control signals. The reference signal is a fixed function. The performance index depends on the differences between reference signal and plant output at the sampling instants. The plant output at any instant depends on its initial value and the history of the control signal up to the instant in question. Therefore, the only available way of lowering the value of the performance index is by changing the control signal. This does not mean just changing its value during a given sampling period, but changing the entire control signal sequence. This might mean only a change during one sampling period, but, in general, would mean changes during many or all sampling periods. The first minimization will consist of finding the sequence of control signals such that the performance index assumes a minimum value and specification of a means of generating this sequence of control signals. This will be accomplished under the assumption that the reference signal is sinusoidal. The need for this restriction will become apparent later.

Determination of the particular function or functions which yield an extremal value for a functional is the subject of the calculus of variations. Bellman[13] has originated a method for solving variational problems. The formalism of this method leads to a functional equation. The functional equation permits finding the extremal trajectory and is a partial differential equation obtained in terms of the limit of an increment along the extremal trajectory. In the discrete case no limit is necessary, and Bellman's method, called dynamic programming, is naturally suited to discrete variational problems such as determination of the control signal for minimum performance index in the present problem. This minimizing control signal will be referred to as the optimal control signal.

Dynamic programming has its basis in the principle of optimality which states that "an optimal policy has the property that whatever the initial system state and initial decision are the remaining decisions must constitute an optimal policy with regard to the system state resulting from the first decision." The terminology used in this statement reflects some of the early applications of the method of dynamic programming which were of a business nature. In the context of the present problem the phrase "optimal policy" means optimal control signal and the term decision indicates a selection of the control signal. A simple example of the application of the principle of optimality will serve to better establish the definitions of some of the terms used above and the mechanics of the dynamic programming method. Also it can be shown that the dynamic programming method is especially well suited for use in discrete problems and why it is advantageous to use a performance index which depends on errors over an infinite interval.

For this purpose consider a regulator system in which the reference signal is zero. Let the plant be described by the differential equation

$$\mathring{X} = m$$

where m(t) is the control signal. Let the performance index be given by

$$S = \int_{t_o}^{T} (X^2 + m^2)\, dt$$

It is required to find m(t) such that S = S[m(t)] is minimized. Because of the minimization, the minimum value of S does not depend on m(t). That is the minimum value of S is the value of S "at" the minimizing m(t). It depends only on the initial state of the plant, $X(t_o)$, and on the time when the integration begins. Thus

$$S_{min} = f[X(t_o),\ t_o]$$

Careful consideration will show that the following is a mathematical statement of the principle of optimality.

$$f[X(t_o),t_o] = \min_m \left\{ \int_{t_o}^{t_o+\Delta t} (X^2 + m^2)dt + f[X(t_o+\Delta t), t_o + \Delta t] \right\}$$

Letting $\Delta t \rightarrow 0$ this becomes

$$f[X(t_o),t_o] = \min_m \left\{ [X^2(t_o) + m^2(t_o)]\Delta t + f[X(t_o),t_o] + \frac{\partial f}{\partial X}\bigg|_{t=t_o} m\Delta t + \frac{\partial f}{\partial t}\bigg|_{t=t_o} \Delta t \right\}$$

where the substitutions $\Delta X = \dot{X}\ \Delta t$ and $\dot{X} = m$ have been used. After some manipulation it can be shown that in the limit as $\Delta t \rightarrow 0$

$$-\frac{\partial f}{\partial t}\bigg|_{t=t_o} = \min_m \left\{ X^2(t_o) + m^2(t_o) + m(t_o)\ \frac{\partial f}{\partial X}\bigg|_{t=t_o} \right\} \tag{19}$$

The minimization operation leads to

$$m(t_o) = -\frac{1}{2}\frac{\partial f}{\partial X}\bigg|_{t=t_o} \tag{20}$$

Thus the partial differential equation in the functional $f(x,t)$

$$-\frac{\partial f}{\partial t} = X^2 - \frac{1}{4}\left(\frac{\partial f}{\partial X}\right)^2 \tag{21}$$

must be solved for $f(x,t)$. Then $m(t_o)$ can be determined from (20). Since the starting time is arbitrary the zero subscript can be dropped. The boundary condition is furnished by the fact that $f(x,T)$ must be zero.

Assuming that

$$f(X,t) = a_o(t) + a_1(t)X + a_2(t)\ X^2$$

and substituting into (21) leads to a set of non-linear ordinary differential equations which may be solved for $a_o(t)$, $a_1(t)$, and $a_2(t)$. The optimum control signal can then be found from (20) and this signal can be generated by a linear time varying feedback structure.

Two things should be noticed from this example. First, had X and m been discrete variables, the limit process could have been skipped. No other change in the formal procedure would be necessary. Second, had the infinite interval $[t,\infty)$ been used instead of the finite interval $(t, T]$, then minimum value of the performance index would depend only on the initial value of X. That is, $X(t_o)$. In this case the functional equation has no time derivatives and hence need not be solved by separation of variables. The result is that the feedback structure would not vary with time. Advantage will be taken of both of these phenomena in the next section.

The First Minimization

Before beginning the development of the first minimization one important point should be discussed. It was stated in Chapter III that the form assumed for a model of the control signal would be verified later. In the above example a form was assumed for the minimum value of the performance index. Both of these assumptions amount to the same thing in two different problems. The situation is similar to that encountered in the solution of partial differential equations by the separation of variables technique. There it is assumed that a function of two variables can be represented as the product of two functions of a single variable. The partial differential equation can then be written as two ordinary differential equations which can be solved for the two assumed functions. Determination of these two functions confirms the validity of the original assumption that the function of two variables could be expressed as the product of two functions each depending on a single variable. The assumption of the form of the minimum value of the performance index in the above example is basically the same. A form is assumed and successful computation of the parameters, a_0, a_1, a_2, verifies the validity of the assumption. To be mathematically rigorous uniqueness arguments would be required. Such arguments applicable to the present problem can be found in publications by Bellman.[14] In the present problem assumption of a form for the control signal dictates the form of the performance index. Determination of the parameters in the expression for the performance index is taken to verify the assumed form of the control signal since the performance index is unique.

The performance index given by equation (18) is

$$S = \sum_{j=o}^{\infty} \left\{ \underline{Z}_j{}' Q \, \underline{Z}_j + \underline{m}_j{}' H \, \underline{m}_j \right\}$$

Using the models developed in Chapter III it is possible to obtain the following relationship.

$$\underline{Z}_{j+1} = \varphi \, \underline{Z}_j + \Delta \, \underline{m}_j \qquad (22)$$

where

$$\varphi = \begin{bmatrix} \varphi_p & 0 \\ 0 & \varphi_j \end{bmatrix}, \quad \Delta = \begin{bmatrix} \Delta_p \\ 0 \end{bmatrix}.$$

For future reference note that the last two rows of Δ are filled with zeros and that the last two rows and last two columns of φ, except for the matrix φ_j, are filled with zeros. The solution of equation (22) is found by induction to be

$$\underline{Z}_j = \varphi^j \underline{Z}_o + \sum_{j=o}^{j-1} \varphi^{j-j-1} \Delta \underline{m}_j \tag{23}$$

which clearly shows that the value of the performance index is a functional of the control sequence $\{\underline{m}_j\}$. Note that equation (22), which conceptually permits evaluation of the performance index, would not be available without an expression of the form

$$\underline{r}_{k+1} = \varphi_j \underline{r}_k$$

for the reference signal. This is the reason for considering sinusoidal reference signals.

Let it be assumed that the form of the optimum control signal will be

$$\underline{m}_{k+1} = \varphi_D \underline{m}_k + \Delta_D \underline{Z}_k \tag{24}$$

Under this assumption the cost, viewing the performance index as a cost function, picked up at the $k^{\underline{th}}$ sampling instant can be determined in terms of quantities existing at the $k\text{-}1^{\underline{st}}$ sampling instant. That is,

$$\underline{Z}_k'Q\,\underline{Z}_k + \underline{m}_k'H\,\underline{m}_k = (\underline{Z}_{k-1}'\varphi' + \underline{m}_{k-1}'\Delta')Q(\varphi\underline{Z}_{k-1} + \Delta\,\underline{m}_{k-1})$$
$$+ (\underline{m}_{k-1}'\varphi_D' + \underline{Z}_{k-1}'\Delta_D')H(\varphi_D\underline{m}_{k-1} + \Delta_D\underline{Z}_{k-1})$$

This expression is of the form

$$\underline{Z}_{k-1}'A_1\underline{Z}_{k-1} + \underline{m}_{k-1}'B_1\underline{m}_{k-1} + \underline{m}_{k-1}'C_1\underline{Z}_{k-1} + \underline{Z}_{k-1}'C_1'\underline{m}_{k-1}$$

where A_1, B_1 and C_1 are symmetric if Q and H are sy mmetric. In like manner it can be shown that

$$\underline{Z}_{k-1}'(A_1 + Q)\underline{Z}_{k-1} + \underline{m}_{k-1}'(B_1 + H) + \underline{m}_{k-1}'C_1\underline{Z}_{k-1} + \underline{Z}_{k-1}'C_1'\underline{m}_{k-1}$$
$$= Z_{k-2}'A_2\underline{Z}_{k-2} + \underline{m}_{k-2}'B_2\underline{m}_{k-2} + \underline{m}_{k-2}'C_2\underline{Z}_{k-2} + \underline{Z}_{k-2}' + \underline{Z}_{k-2}C_2'\underline{m}_{k-2}\,.$$

where A_2, B_2, and C_2 are symmetric if Q and H are symmetric. This expression gives the cost accumulated at the $k^{\underline{th}}$ and $k-1^{\underline{st}}$ sampling instants in terms of the values of the variables at the $k-2^{\underline{nd}}$ sampling instant. By induction it can be shown that the value of the performance index depends only on the initial values of the variables and must be of the form

$$S_{min} = S_{min}(\underline{Z}_o, \underline{m}_o)$$
$$= \underline{Z}_o'\,\mathcal{Q}\,\underline{Z}_o + \underline{m}_o'\mathcal{H}\,\underline{m}_o + \underline{m}_o'\mathcal{K}\underline{Z}_o' + \underline{Z}_o'\mathcal{K}'\underline{m}_o \qquad (25)$$

The subscript "min" indicates that the minimizing control signal sequence has been used. Since the initial instant is arbitrary the subscript zero in equation (25) can be changed to j.

A mathematical statement of the principle of optimality would be the following.

$$S_{min}(\underline{Z}_{j-1}, \underline{m}_{j-1}) = \min_{\underline{m}_j} \left\{ \underline{Z}'_{j-1} Q \underline{Z}_{j-1} + \underline{m}'_{j-1} H \underline{m}_{j-1} + S_{min}(\underline{Z}_j, \underline{m}_j) \right\} \tag{26}$$

S_{min} will be replaced in both places it appears in (26) by the expression of equation (25) with appropriate subscript changes. The result is

$$\underline{Z}'_{j-1} \mathcal{Q} \underline{Z}_{j-1} + \underline{m}'_{j-1} \mathcal{H} \underline{m}_{j-1} + m'_{j-1} \mathcal{K} \underline{Z}_{j-1} + \underline{Z}'_{j-1} \mathcal{K}' \underline{m}_{j-1} =$$

$$\min_{m_j} \underline{Z}'_j Q \underline{Z}_j + \underline{m}'_j H \underline{m}_j + \underline{Z}'_j \mathcal{Q} \underline{Z}_j + \underline{m}'_j \mathcal{H} \underline{m}_j + \underline{m}'_j \mathcal{K} \underline{Z}_j + \underline{Z}'_j \mathcal{K} \underline{m}_j \tag{27}$$

Performing the minimization operation indicates that

$$H\underline{m}_j + (\underline{m}'_j H)' + \mathcal{H} \underline{m}_j + (\underline{m}'_j \mathcal{H})' + \mathcal{K} \underline{m}_j + (m'_j \mathcal{K})' = 0$$

This reduces to

$$(H + H' + \mathcal{H} + \mathcal{H}')\underline{m}_j = - 2\mathcal{K} \underline{Z}_j$$

which defines the optimum $j^{\underline{th}}$ control signal as

$$\underline{m}_j = - (H + \mathcal{H})^{-1} \mathcal{K} \underline{Z}_j = -(H + \mathcal{H})^{-1} \mathcal{K} (\varphi \underline{Z}_{j-1} + \Delta \underline{m}_{j-1}) \tag{28}$$

Equation (28) may be identified with equation (24) to obtain

$$\varphi_D = -(H + \mathcal{H})^{-1} \mathcal{K} \Delta$$

$$\Delta_D = -(H + \mathcal{H})^{-1} \mathcal{K} \varphi$$

Determination of φ_D and Δ_D, and hence the optimum tracking system, will be possible provided $\mathcal{H}$ and $\mathcal{K}$ can be determined.

In order to determine the matrices $\mathcal{Q}$, $\mathcal{H}$, and $\mathcal{K}$, equation (27) will be rewritten with all variab les on the right side referred back one sampling instant to the j-1$\underline{\text{th}}$ instant and $\underline{m}_j$ replaced by its optimum value.

$$\underline{Z}'_{j-1}\mathcal{Q}\,\underline{Z}_{j-1} + \underline{m}'_{j-1}\mathcal{H}\,\underline{m}_{j-1} + \underline{m}'_{j-1}\mathcal{K}\,\underline{Z}_{j-1} + \underline{Z}'_{j-1}\mathcal{K}'\underline{m}_{j-1}$$

$$= (\underline{Z}'_{j-1}\,\varphi' + \underline{m}'_{j-1}\,\Delta')(Q + \quad)(\varphi\underline{Z}_{j-1} + \Delta\underline{m}_{j-1}) + (\underline{m}'_{j-1}\varphi'_D +$$

$$+ \underline{Z}'_{j-1}\Delta'_D)(H + \mathcal{H})(\varphi_D\underline{m}_{j-1} + \Delta\underline{m}_{j-1} + \Delta_D\underline{Z}_{j-1})$$

$$+ (\underline{m}'_{j-1}\,\varphi'_D + \underline{Z}'_{j-1}\,\Delta'_D)(\mathcal{K})(\varphi\underline{Z}_{j-1} + \Delta\underline{m}_{j-1})$$

$$+ (\underline{Z}'_{j-1}\varphi' + \underline{m}'_{j-1}\Delta')\,\mathcal{K}'(\varphi_D m_{j-1} + \Delta_D\underline{Z}_{j-1}) \qquad (24)$$

Since equation (29) must be satisfied for any $\underline{Z}_{j-1}$ and $\underline{m}_{j-1}$, like coefficients may be equated. The result is

$$\mathcal{Q} = \varphi'(\mathcal{Q} + Q)\varphi + \Delta'_D(H + \mathcal{H})\Delta_D + \Delta'_D\mathcal{K}\varphi + \varphi'\mathcal{K}'\Delta_D$$

$$\mathcal{H} = \Delta'(Q + \mathcal{Q})\Delta + \varphi'_D(H + \mathcal{H})\varphi_D + \varphi'_D\mathcal{K}\Delta + \Delta'\mathcal{K}'\varphi_D$$

$$\mathcal{K} = \Delta'(Q + \mathcal{Q})\varphi + \varphi'_D(H + \mathcal{H})\Delta_D + \varphi'_D\mathcal{K}\varphi + \Delta'\mathcal{K}\,\Delta_D$$

Making the substitutions

$$\mathcal{K}\varphi = -\,(H + \mathcal{H})\Delta_D$$

$$\mathcal{K}\Delta = -\,(H + \mathcal{H})\varphi_D$$

these equations reduce to

$$\mathcal{Q} = \varphi'(Q + \mathcal{Q})\varphi - \varphi'\mathcal{K}'(H + \mathcal{H})^{-1}\mathcal{K}\varphi$$

$$\mathcal{H} = \Delta'(Q + \mathcal{Q})\Delta - \Delta'\mathcal{K}'(H + \mathcal{H})^{-1}\mathcal{K}\Delta \qquad (30)$$

$$\mathcal{K} = \Delta'(Q + \mathcal{Q})\varphi - \Delta'\mathcal{K}'(H + \mathcal{H})^{-1}\mathcal{K}\varphi$$

This is a set of three non-linear matrix equations in three unknown matrices. The solution may be simplified by defining a matrix M as

$$M = Q + \mathcal{Q} - \mathcal{K}'(H + \mathcal{H})^{-1}\mathcal{K} \qquad (31)$$

In terms of M, the three matrices $\mathcal{Q}$, $\mathcal{H}$, and $\mathcal{K}$ are

$$\mathcal{Q} = \varphi' M\varphi$$

$$\mathcal{H} = \Delta' M\Delta \qquad (32)$$

$$\mathcal{K} = \Delta' M\varphi$$

substituting equations (32) into equation (31) yields

$$M = Q + \varphi' M\varphi - \varphi' M'\Delta(H + \Delta' M\Delta)^{-1}\Delta' M\varphi \qquad (33)$$

which is a single matrix equation in a single unknown matrix M. Determination of M leads to values of $\mathcal{Q}$, $\mathcal{H}$, and $\mathcal{K}$ and therefore specifies optimum system structure via equation (28).

In an application of this synthesis method, solution of equation (33) would likely be the most difficult step. Hand solution would, in general, be impractical if not impossible. However, it is possible to solve this equation in many cases with the aid of a digital computer. The method which has been used in the example problems to be presented in Chapter VI consists of assuming an M, substituting this M into the right hand side of equation (33), and computing a new M which in turn is used to find another new M. The process is repeated until convergence occurs. This method has worked in all examples tried. Equation (33) really amounts to a set of simultaneous non-linear algebraic equations. There exist few sets of sufficient conditions for the existence of solutions of such sets of equations. The difficulty of application of existing theorems is compounded by the fact that equation (33) is written in terms of matrices. Most theorems require that the scalar equations be known. Since the size of M depends on the order of the plant it is not feasible to try to write the set of algebraic equations represented by equation (33) for any general case.

It can be shown that the feedback structure of the optimum tracking system as determined above does not depend on the frequency of the reference signal. In order to show this first note that H does not depend on any element of M which is in either the last two rows or either of the last two columns. This is true because the last two rows of Δ are filled with zeros. This independence is easily seen by writing the expression for $\mathcal{H}$ from equation

$$\mathcal{H} = \begin{bmatrix} \cdot & \cdot & \cdot & 0 & 0 \\ & & & \vdots & \vdots \\ \cdot & \cdot & \cdot & 0 & 0 \end{bmatrix} \begin{bmatrix} m_{11} & \cdots & m_{1n+1} & m_{1n+2} \\ \vdots & & \vdots & \vdots \\ m_{n+11} & \cdots & m_{n+1n+1} & \vdots \\ m_{n+21} & \cdots & & m_{n+2n+2} \end{bmatrix} \begin{bmatrix} \vdots & & \vdots \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{bmatrix}$$

Also note that $\mathcal{H}$ does not depend on any element of M which is in the intersection of the last two rows and the last two columns. Due to the symmetry of M, $\mathcal{H}$ will depend on elements in the last two rows and columns of M which are not in this intersection. Again this independence can be seen by writing out $\mathcal{H}$ from equation (32).

$$\mathcal{H} = \begin{bmatrix} \cdots & 0 & 0 \\ & \vdots & \vdots \\ \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} m_{11} & \cdots & m_{1n+1} & m_{1n+2} \\ \vdots & & \vdots & \vdots \\ m_{n+11} & \cdots & m_{n+1n+1} & \vdots \\ m_{n+12} & \cdots & & m_{n+2n+2} \end{bmatrix} \begin{bmatrix} \varphi & 0 \\ & \\ 0 & \varphi_j \end{bmatrix} \tag{34}$$

From equation (34) it is seen that the last two rows of $\mathcal{H}$ will depend on frequency because of the term φ_j. However, these terms influence input gains only. If it can be shown that upper left n x n submatrix of M does not change with reference signal frequency then it can be said that the feedback structure is fixed. For this purpose consider the solution of equation (33) by the recomputation technique previously outlined. Suppose a solution has been obtained and the reference signal frequency is then changed and a new solution sought. As a starting point the old solution will be used. When this is used to compute the last term of equation (33) no terms from the lower right 2 x 2 submatrix of M appear. This is because everywhere M appears, Δ also appears and does away with either the last two rows or the last two columns of M. Also reference signal frequency will appear only in the lower right 2 x 2 submatrix of this term. Hence the upper left n x n submatrix of the last term is the same as with the previous frequency. Likewise the upper left n x n submatrix of the term $\varphi' M \varphi$ will not change with a change in frequency. Hence the new M matrix has the same upper right n x n submatrix as it did with the previous reference signal frequency. This proves that optimum system feedback structure does not change with reference signal frequency.

The Second Minimization

The system determined by the first minimization is for a reference signal of a specific frequency and the control element can be represented by an expression of the form

$$\underline{m}_{k+1} = \Phi_D \underline{m}_k + \Delta_{D1} \underline{X}_k + \Delta_{D2} \underline{r}_k \tag{35}$$

where only the elements of Δ_{D2} vary with changes in the reference signal frequency. Since the elements of $\underline{r}_k$ are $r(t)$, the reference signal, and $\dot{r}(t)$, its derivative, the term involving the input can be written

$$\Delta_{D2}\underline{r}_k = \begin{bmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \\ \vdots & \vdots \\ \delta_{m1} & \delta_{m2} \end{bmatrix} \begin{bmatrix} \sin \omega kT \\ \omega \cos \omega kT \end{bmatrix} = \begin{bmatrix} \delta_{11} \sin \omega kT + \delta_{12}\, \omega \cos \omega kT \\ \\ \\ \delta_{n1} \sin \omega kT + \delta_{n2} \omega \cos \omega kT \end{bmatrix} = \begin{bmatrix} A_1 \sin(\omega kT + \varphi_1) \\ \\ \\ A_m \sin(\omega kT + \varphi_m) \end{bmatrix}$$

where

$$A_1 = \sqrt{\delta_n^{\,2} + \omega^2 \delta_{12}^{\,2}}\ ,\ \varphi_1 = \tan^{-1} \frac{\delta_{12}\omega}{\delta_{11}}\ ,\ \text{etc.}$$

Now, from the first minimization all elements of Δ_{D2}, referred to as input gains, are known as a function of frequency. From this knowledge A_1, φ_1, A_2, φ_2, etc. can be determined as functions of frequency. If only one certain frequency is to be applied to the system then there is no question about what input gains to use. However, if a reference signal having a continuous spectrum is to be applied some method must be used for selecting a set of input gains to be used.

This selection is made by a second minimization process. The performance index to be minimized in this case depends on the deviation of the system from the ideal, as determined by the first minimization, at several evenly spaced frequencies in the band containing the reference signal. The error at each of these frequencies is weighted according to the frequency distribution of reference signal energy. The performance index is composed of a phase dependent term and an amplitude dependent term. The amplitude term is

$$I_1 = \sum_{j=1}^{N} \left[\left(\sqrt{\delta_{11}^{2}(\omega_j) + \omega_j^{2}\delta_{12}^{2}(\omega_j)} - \sqrt{G_{11}^2 + \omega_j^{2} G_{12}^{2}} \right)^2 S(\omega_j) \right]$$

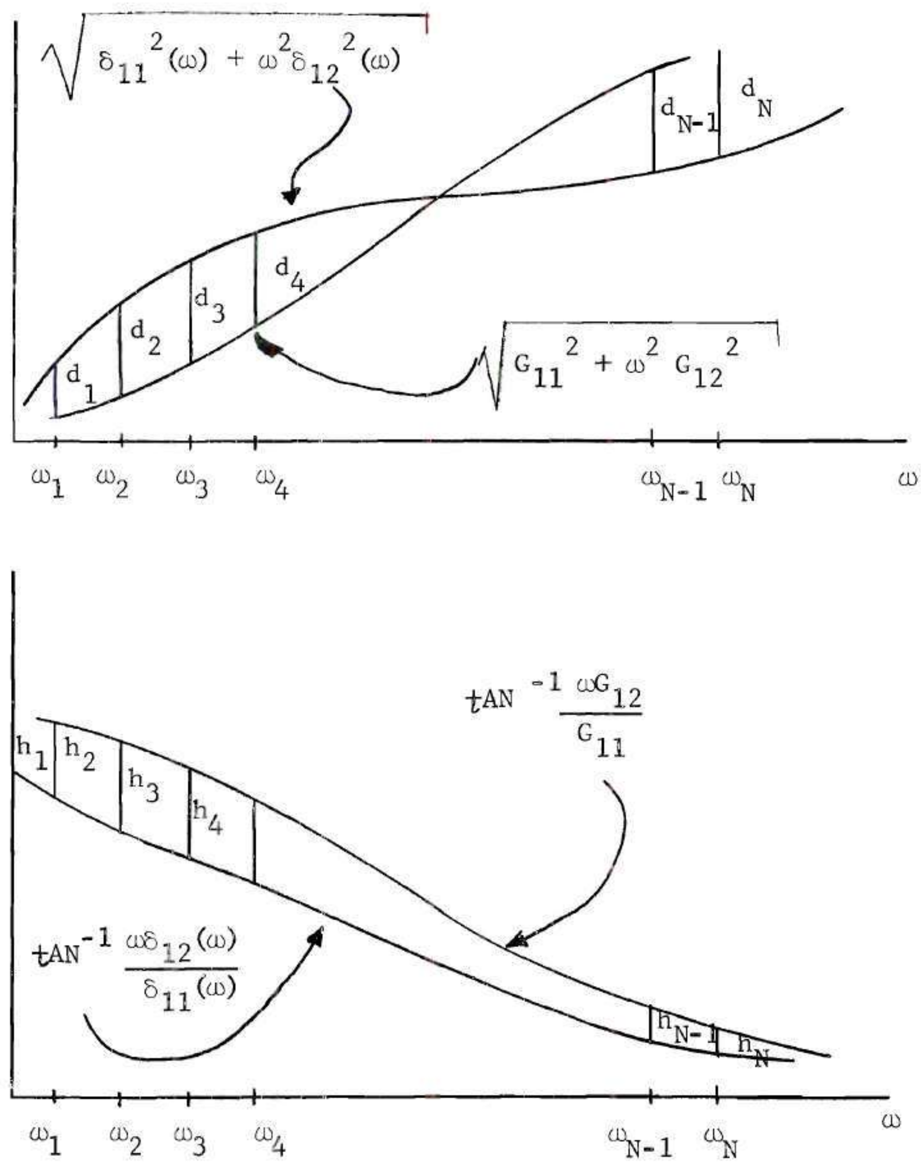
where G_{11} and G_{12} are the fixed input gains to be used and $S(\omega_j)$ is the power contained in the reference signal between ω_j and ω_{j+1}. The phase term is

$$I_2 = \sum_{j=1}^{N} \left[\left(\tan^{-1} \frac{\omega_j \delta_{12}(\omega_j)}{\delta_{11}(\omega_j)} - \tan^{-1} \frac{\omega_j G_{12}}{G_{11}} \right)^2 S(\omega_j) \right]$$

The actual performance index minimized is

$$I = I_1 + \lambda I_2 \tag{36}$$

where λ is a parameter which allows arbitrary importance to be attached to phase behavior. The terms in this performance index are shown graphically in Figure 3. Equation (35) and the minimization lead to a set of two simultaneous non-linear equations with G_{11}, and G_{12} as unknowns. Note that this process must be repeated for each row of Δ_{D2}. Specification



$$I = \sum_{j=1}^{N} (d_j^{\,2} + \lambda h_j^{\,2})$$

Figure 3. Illustration of Performance Indices for Second Minimization.

of the elements of Δ_{D2} is the final step in the synthesis procedure.

An Alternate Approach to the Second Minimization

The frequency response design method has proved to be among the most useful classical design techniques. The usefulness of this technique has not been exploited by modern synthesis methods. These latter methods are, in general, based on minimization of a time domain performance index and require that system mathematical models be time domain models. All operations are carried out in the time domain and no opportunity presents itself for effective utilization of frequency domain techniques. Because of the approach taken to the tracking system synthesis problem here it is not only possible, but also advantageous, to utilize system frequency response for selection of system input gains. In particular, a method will be developed for selecting input gains so that the overall system has, as nearly as possible, unity gain and constant time delay over the band of frequencies containing the reference signal. In order to do this the problem of computing system frequency response from the vector difference equation which models the system must be investigated. The purpose of this section is to make this investigation and to present the above mentioned alternate approach to selecting system input gains.

System Frequency Response

The complete system is modeled by the linear difference equations (13). It is well known[15] that a set of stable linear difference equations when forced by a sinusoid respond sinusoidally in the steady state at the same frequency as the forcing function. In other words, if $\underline{r}_k$ in equation (13) has sinusoidal components of frequency ω then, in the steady state,

$\underline{X}_k$ and $\underline{m}_k$ will have sinusoidal components of the same frequency ω. Therefore it is only necessary to compute the amplitude and phase angle of each component of $\underline{X}_k$ and $\underline{m}_k$ when each component of $\underline{r}_k$ has a known amplitude and phase angle. This process, repeated for several reference signal frequencies, determines the frequency response of the system.

Determination of the required amplitudes and phase angles can be formulated as an algebraic problem. If the reference signal is sinusoidal it can be represented as

$$r_1 = r(t) = \cos \omega t$$

Also

$$r_2 = \dot{r}(t) = \omega \sin \omega t$$

The substitution

$$r_1 = \mathrm{Re}\left\{e^{j\omega t}\right\}$$

is useful. In terms of this

$$r_2 = \mathrm{Re}\left\{j\omega\ e^{j\omega t}\right\}$$

where Re is read "real part of." For example Re $\{x + jy\}$ equals X. The "real part of" operator may be dropped with the understanding that when a solution of equation (13) is obtained, the actual system variables will correspond to the real part of the obtained solution. Therefore

$$r_{1k} = e^{j\omega kT}$$

$$r_{2k} = j\omega e^{j\omega kT}$$

and

$$\underline{r}_k = \begin{bmatrix} 1 \\ \\ j\omega \end{bmatrix} e^{j\omega kT}$$

It has already been argued that $\underline{X}_k$ and $\underline{m}_k$ will have, in the steady state, sinusoidal components of the same frequency as the reference signal. They will, in general, have different amplitudes and phase angles. Therefore

$$\begin{bmatrix} \underline{X}_k \\ \\ \underline{m}_k \end{bmatrix} = \begin{bmatrix} c_1 e^{j(\omega kT + \theta_1)} \\ \vdots \\ c_j e^{j(\omega kT + \theta_j)} \end{bmatrix}$$

where j is the number of components of $\underline{X}_k$ plus the number of components of $\underline{m}_k$. Ths $\underline{c}$'s and the $\underline{\theta}$'s are the amplitudes and phase angles, respectively, of the system variables in response to unit amplitude and zero phase angle of the input r(t).

Defining a new vector $\underline{y}_k$ to be

$$\underline{y}_k = \begin{bmatrix} \underline{X}_k \\ \\ \underline{m}_k \end{bmatrix} = \begin{bmatrix} c_1 e^{j\theta_1} \; e^{j\omega kT} \\ \\ c_j e^{j\theta_j} \; e^{j\omega kT} \end{bmatrix} = \begin{bmatrix} c_1 e^{j\theta_1} \\ \\ c_j e^{j\theta_j} \end{bmatrix} e^{j\omega kT} \tag{37}$$

equation (13) becomes

$$\underline{y}_{k+1} = \Lambda \, \underline{y}_k + \Omega \begin{bmatrix} 1 \\ \\ j\omega \end{bmatrix} e^{j\omega kT}$$

or

$$\begin{bmatrix} c_1 e^{j\theta_1} \\ \\ c_j e^{j\theta_j} \end{bmatrix} e^{j\omega(k+1)T} = \Lambda \begin{bmatrix} c_1 e^{j\theta_1} \\ \\ c_j e^{j\theta_j} \end{bmatrix} e^{j\omega kT} + \Omega \begin{bmatrix} 1 \\ \\ j\omega \end{bmatrix} e^{j\omega kT} \tag{38}$$

where

$$\Lambda = \begin{bmatrix} \varphi & \Delta \\ \\ \Delta_{D1} & \varphi_D \end{bmatrix}$$

and

$$\Omega = \begin{bmatrix} 0 & 0 \\ \cdot & \cdot \\ \cdot & \cdot \\ & \Delta_{D2} \end{bmatrix}$$

The k variation may be removed from equation (38) by dividing both sides by the factor $e^{j\omega kT}$. The result is

$$\begin{bmatrix} c_1 e^{j\theta_1} \\ \cdot \\ \cdot \\ c_j e^{j\theta_j} \end{bmatrix} e^{j\omega T} = \Lambda \begin{bmatrix} c_1 e^{j\theta_1} \\ \\ c_j e^{j\theta_j} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \cdot & \cdot \\ \cdot & \cdot \\ & \Delta_{D2} \end{bmatrix} \begin{bmatrix} 1 \\ \\ j\omega \end{bmatrix}$$

This equation is a set of j complex equations in j complex unknowns and can be rewritten and solved as follows.

$$(e^{j\omega T} I - \Lambda) \begin{bmatrix} c_1 e^{j\theta_1} \\ \\ c_j e^{j\theta_j} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \cdot & \cdot \\ \cdot & \cdot \\ & \Delta_{D2} \end{bmatrix} \begin{bmatrix} 1 \\ \\ j\omega \end{bmatrix}$$

where I is the unit diagonal or identity matrix.

If

$$(e^{j\omega T}\, I - \Lambda)^{-1}$$

exists then

$$\begin{bmatrix} c_1 e^{j\theta_1} \\ \vdots \\ c_j e^{j\theta_j} \end{bmatrix} = (e^{j\omega T} I - \Lambda)^{-1} \begin{bmatrix} 0 & 0 \\ \vdots & \vdots \\ \multicolumn{2}{c}{\Delta_{D2}} \end{bmatrix} \begin{bmatrix} 1 \\ j\omega \end{bmatrix} \tag{39}$$

Since X_{1k}, the first component of $\underline{X}_k$, is the system output variable it is only necessary to solve equation (39) for $c_1 e^{j\theta_1}$ for several values of ω to determine the frequency response of the system.

Selection of Input Gains

The symbol Δ_{D2} represents a matrix having two columns and the same number of rows as the control signal vector. This matrix contains the input gains, the fixed values of which were symbolized G_{11}, G_{12}, etc. in the last section. In terms of this symbolism equation (39) becomes

$$\begin{bmatrix} c_1 e^{j\theta_1} \\ \\ c_j e^{j\theta_j} \end{bmatrix} = (e^{j\omega T} I - \Lambda)^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ G_{11} + j\omega\, G_{12} \end{bmatrix} \tag{40}$$

where, for purposes of illustration, it has been assumed that the control signal is a scalar. Note that, since the control signal is one dimensional and since only $c_1 e^{j\theta_1}$ is to be determined, it is necessary to compute only one element of

$$(e^{j\omega T}\ I - \Lambda)^{-1}$$

In computing this element it is necessary to compute

$$\mathrm{Det}\ |e^{j\omega T}I - \Lambda|$$

Some effort can generally be saved by noting that

$$\mathrm{Det}\ |e^{j\omega T}I - \Lambda| = e^{j\omega jT} + a_1 e^{j\omega(j-1)T} + \dots + a_{j-1}e^{j\omega T} + a_j$$

which is the characteristic equation of the matrix Λ. It can be shown[16] that the coefficients of this equation are given by

$$a_1 = -\,\mathrm{Tr}(\Lambda)$$

$$a_2 = -\tfrac{1}{2}\,[a_1\mathrm{Tr}(\Lambda) + \mathrm{Tr}(\Lambda^2)]$$

$$\vdots$$

$$a_j = -\tfrac{1}{j}\,[a_{j-1}\mathrm{Tr}(\Lambda) + a_{j-2}\mathrm{Tr}(\Lambda^2) + \dots + a_1\mathrm{Tr}(\Lambda^{j-1}) + \mathrm{Tr}(\Lambda^j)]$$

where $\mathrm{Tr}(\Lambda)$ is called the trace of Λ and is the sum of the elements on the main diagonal of Λ.

After the necessary element of the inverse matrix in equation (40) has been determined it is possible to express $c_1 e^{j\theta_1}$ as

$$c_1 e^{j\theta_1} = F(j\omega)[G_{11} + j\omega G_{12}] \tag{41}$$

Equation (41) yields the frequency response of the system described by equation (13). The deviation of system fre quency response from flat response can be gauged by

$$J_1 = \sum_{j=1}^{N} \left\{ \left| F(j\omega_j)[G_{11} + j\omega_j G_{12}] \right| - 1 \right\}^2 S(\omega_j)$$

where the N frequencies are spaced uniformly over the band containing the reference signal. Again $S(\omega_j)$ is a weighting factor which depends on the amount of energy in the reference signal between ω_j and ω_{j+1}. The deviation from constant time delay can be expressed as

$$J_2 = \sum_{j=1}^{N} \left\{ \left[\alpha(\omega_j) + \tan^{-1} \frac{\omega_j G_{12}}{G_{11}} - K\omega_j \right]^2 S(\omega_j) \right\}$$

where $\alpha(\omega_j)$ is the angle of $F(j\omega_j)$, and K can be any real number. Different values of K indicate different values of time delay. Determination of G_{11} and G_{12} such that

$$J = J_1 + J_2$$

is minimized yields the values of G_{11} and G_{12} which cause the system to have, as nearly as possible in the mean square sense, flat frequency response and constant time delay. Such values of G_{11} and G_{12} can be found using standard calculus techniques.

CHAPTER V

SENSITIVITY ANALYSES OF THE SYNTHESIZED SYSTEM

This chapter contains an analysis of system performance sensitivity to quantization associated with the digital control element, a discussion and mathematical treatment of the sensitivity of system performance to changes in sampling period length, and, finally, an investigation of system stability which leads to a set of sufficient conditions for system stability.

Quantization

In order to obtain a measure of system performance sensitivity to quantization associated with the digital control element, two quantities will be determined. First, a bound on the tracking error due to quantization will be determined. This error bound will be conservative in that it will be the maximum possible error which could exist. Second, the mean square tracking error will be determined.

Each variable which is either fed into or fed out of the control element will be quantized as part of the analog to digital or digital to analog conversion process. Thus when one of the system variables is sampled its value may either be increased or decreased by an amount not exceeding some positive number, say h, which describes the quantization. That is, when a variable is converted to digital form it need never be changed by more than h units. The spacing between permissible digital values which a variable may assume is 2h. For analysis purposes this quantization process will be modeled by a control element with no quantization, but with a separate error signal added to each variable going into or out of the control element. These extra error signals represent the quantization errors and are such that the variable plus the corresponding error signal equals the quantized value of the variable. Thus it is seen that the error signals, any of them and at any sampling instant, will be assumed to be equally likely to have any value between -h and h. A pictorial representation is given in Figure 4.

The operation performed by the control element is known. With quantization present, the control element operates on the quantization errors in the same way as it does the system variables. Thus the control element is described by the equation

$$\underline{m}_{k+1} = \varphi_D \underline{m}_k + \Delta_{D1}(\underline{X}_k + \underline{\varepsilon}_{xk}) + \Delta_{D2}\underline{\varepsilon}_{rk} \tag{42}$$

where the components of $\underline{\varepsilon}_{xk}$ and $\underline{\varepsilon}_{rk}$ represent the quantization errors incurred by the components of $\underline{X}_k$ and $\underline{r}_k$ respectively. Thus

$$\underline{\varepsilon}_{xk} = \begin{bmatrix} \varepsilon_{x_1 k} \\ \cdot \\ \cdot \\ \cdot \\ \varepsilon_{x_n k} \end{bmatrix} \qquad \underline{\varepsilon}_{rk} = \begin{bmatrix} \varepsilon_{r_1 k} \\ \\ \varepsilon_{r_2 k} \end{bmatrix}$$

In writing this equation no allowance has been made for quantization of the components of $\underline{m}_k$ as they are fed back to form part of $\underline{m}_{k+1}$. This effectively assumes that the feedback is internal to the control element and therefore that the components of $\underline{m}_k$ can be kept in digital form. This being true, no quantization of the components of $\underline{m}_k$ occurs in the

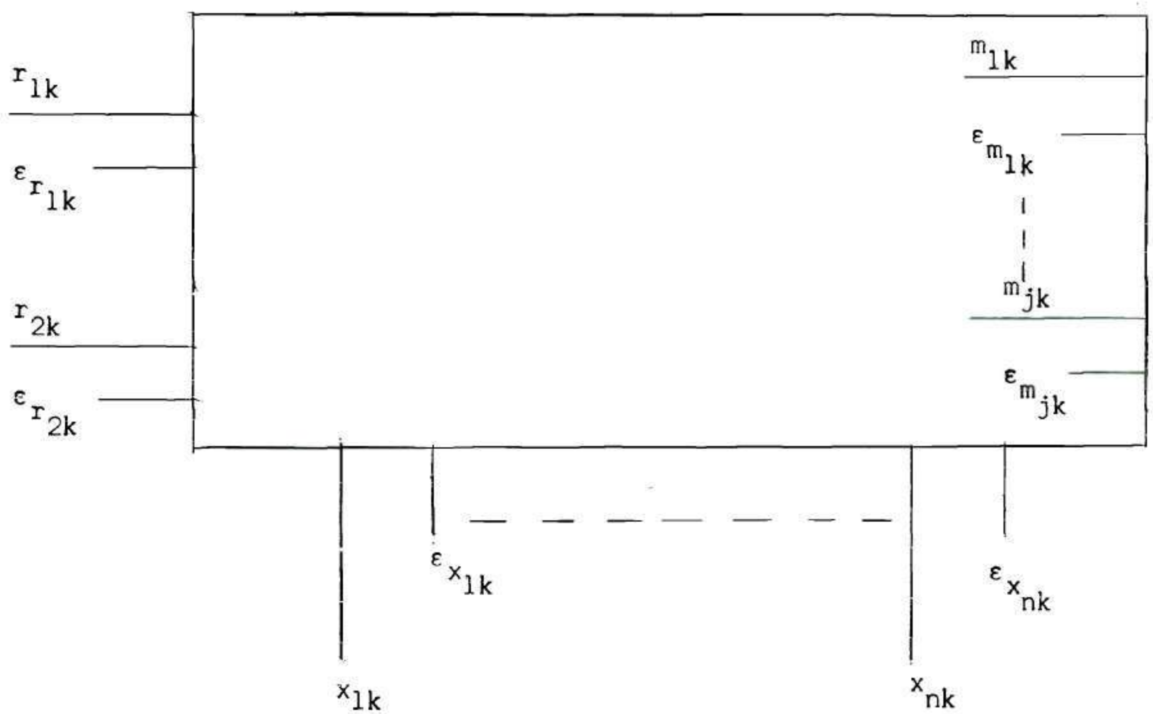


Figure 4. Block Diagram of Control Element Showing Quantization Errors as Separate Inputs and Outputs.

feedback process. Also the input or reference signal, $\underline{r}_k$, has been assumed to be zero. Since the system is linear, superposition holds and the effect of quantization on system performance can be obtained with only the quantization error signals present as system inputs.

When $\underline{m}_k$ is brought out of the control element and applied to the plant input, its components are quantized. The resulting error is modeled as shown in Figure 4. Thus the recursion relationship for the plant state vector is

$$\underline{X}_{k+1} = \varphi\ \underline{X}_k + \Delta(\underline{m}_k + \underline{\varepsilon}_{mk}) \tag{43}$$

where $\underline{\varepsilon}_{mk}$ represents the error due to quantization of the components of $\underline{m}_k$.

Equations (42) and (43) may be written as a single equation as follows.

$$\begin{bmatrix} \underline{X}_{k+1} \\ \underline{m}_{k+1} \end{bmatrix} = \begin{bmatrix} \varphi & \Delta \\ \Delta_{D1} & \varphi_D \end{bmatrix} \begin{bmatrix} \underline{X}_k \\ \underline{m}_k \end{bmatrix} + \begin{bmatrix} 0 & 0 & \Delta \\ \Delta_{D1} & \Delta_{D2} & 0 \end{bmatrix} \begin{bmatrix} \underline{\varepsilon}_{xk} \\ \underline{\varepsilon}_{rk} \\ \underline{\varepsilon}_{mk} \end{bmatrix} \tag{44}$$

The solution of equation (44) gives the response of the system to the quantization error signals. The solution is

$$\begin{bmatrix} \underline{X}_k \\ \underline{m}_k \end{bmatrix} = \sum_{j=1}^{k-1} \Lambda^{k-j-1}\ B\ \underline{\varepsilon}_j \tag{45}$$

where

$$\Lambda = \begin{bmatrix} \varphi & \Delta \\ \\ \Delta_{D1} & \varphi_D \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & \Delta \\ \\ \Delta_{D1} & \Delta_{D2} & 0 \end{bmatrix}, \quad \underline{\varepsilon}_j = \begin{bmatrix} \underline{\varepsilon}_{xj} \\ \underline{\varepsilon}_{rj} \\ \underline{\varepsilon}_{mj} \end{bmatrix}.$$

The transient part of the solution has been dropped since it is not due to quantization. Before continuing the present development it is necessary to state a certain theorem. This is known as Sylvester's theorem and states[16] that if the n eigenvalues of an n x n square matrix M are all distinct and if P(M) is a polynomial in M of the form

$$P(M) = c_o M^k + c_1 M^{k-1} + \ldots + c_{k-1} M + c_k I$$

where I is the identity matrix and the c_j are constants, then the polynomial P(M) may be expressed in the form

$$P(M) = \sum_{r=1}^{n} \lim_{\lambda - \lambda r} \frac{f(\lambda)}{f'(\lambda)} (\lambda I - M)^{-1} P(\lambda)$$

where the λ_r are the eigenvalues of M and $f(\lambda)$ is the characteristic function of M.

$$f(\lambda) = \text{Det} \left| \lambda I - M \right|$$

The function $f(\lambda)$ is the polynomial whose roots are the eigenvalues of M. The symbol $f'(\lambda)$ indicates the derivative of this polynomial with respect to λ.

Application of Sylvester's theorem to equation (45) permits this equation to be rewritten

$$\begin{bmatrix} \underline{X}_k \\ \\ \underline{m}_k \end{bmatrix} = \sum_{j=1}^{k-1} (\lambda_1^{k-j-1} \Lambda_1 B \underline{\varepsilon}_j + \ldots + \lambda_j^{k-j-1} \Lambda_1 B \underline{\varepsilon}_j) \tag{46 }$$

where

$$\Lambda_r = \lim_{\lambda \to \lambda_r} \frac{f(\lambda)(\lambda I - \Lambda)^{-1}}{f'(\lambda)}$$

and

$$f(\lambda) = \mathrm{Det}|\lambda I - \Lambda|$$

The quantities $\lambda_1, \ldots, \lambda_j$ area the eigenvalues of Λ.

Each component of $\underline{\varepsilon}_j$ is bounded in absolute value by h. Equation (46) gives each component of $\underline{X}_k$ and $\underline{m}_k$ as a linear combination of the components of $\underline{\varepsilon}_1$, $\underline{\varepsilon}_2, \ldots\ldots$, $\underline{\varepsilon}_{k-1}$. Therefore it is possible to compute a bound for each element of $\underline{X}_k$ and $\underline{m}_k$ in terms of h and plant and control element parameters from equation (46). Plant and control element parameters enter into the bound through $\Lambda_1, \ldots, \Lambda_j$ and B. It can be concluded that if all of the eigenvalues of Λ are less than one in absolute value then the bound on the $k^{\underline{th}}$ tracking error due to quantization approaches a limit as k approaches infinity. This is true because the geometric series in equation (46) have finite sums if all of the λ_r are less than one in absolute value. The general influence of plant and control element parameters on the bound is not easily exhibited because of the complicated way in which these parameters must be manipulated and combined to obtain Λ_1, $\Lambda_2, \ldots,$ Λ_j, and B. Bounds will be obtained for several systems in the next chapter and from these examples a feel can be developed for how system parameters influence the bounds.

In order to compute the mean square tracking error due to quantization each component of $\underline{\varepsilon}_j$ in equation (46) is regarded as a random variable which is equally likely to have any value b etween -h and h. Under this assumption the variance of the quantization error is

$$\sigma_\varepsilon^{\;2} = \frac{h^2}{3}$$

These random variables which represent quantization error are centered. It is true[17] that if X and Y are independent centered random variables and if Z is their sum then the variance of Z is the variance of X plus the variance of Y. Also if X is a random variable and K is a fixed real number, then KX is a random variable whose variance is K^2 times the variance of X. These facts will be used to compute the mean square tracking error from equation (46).

It was pointed out in connection with determination of a tracking error bound above that equation (46) gives the tracking error as a linear combination of the components of $\varepsilon_1,\ldots,\underline{\varepsilon}_{k-1}$. Thus

$$X_{1k} = \sum_{j=1}^{k-1} \lambda_1^{k-j-1}(\alpha_{111}\varepsilon_{xij} + \alpha_{112}\varepsilon_{x2j} + \ldots + \alpha_{11j}\varepsilon_{x1j})$$

$$+ \sum_{j=1}^{k-1} \lambda_2^{k-j-1}(\alpha_{211}\varepsilon_{x1j} + \alpha_{212}\varepsilon_{x2j} + \ldots + \alpha_{21j}\varepsilon_{xjj})$$

$$+ \ldots + \sum_{j=1}^{k-1} \lambda_j^{k-j-1}(\alpha_{j11}\varepsilon_{x1j} + \alpha_{j12}\varepsilon_{x2j} + \ldots + \alpha_{j1j}\varepsilon_{xij}) \quad (47)$$

where $$\Lambda_{\ell}B = \begin{bmatrix} \alpha_{\ell 11} & \alpha_{\ell 12} & \cdots & \ell_{\alpha 1j} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \alpha_{\ell j1} & \cdot\cdot\cdot\cdot & & \alpha_{\ell jj} \end{bmatrix}$$

In equation (47) each of the sums in parentheses may be considered a random variable. Letting

$$\beta_{ij} = \alpha_{j11}\varepsilon_{x1j} + \ldots + \alpha_{j1j}\varepsilon_{xij} \tag{48}$$

equation (47) becomes

$$X_{1k} = \sum_{j=1}^{k-1} (\lambda_1^{k-j-1}\beta_{1j} + \ldots + \lambda_j^{k-j-1}\beta_{ij})$$

The plant and reference signal state variables are linearly independent. There is no way of predicting one of these variables from knowledge of the others. Since the quantization errors depend on the values of these variables, they must be independent. Under this assumption the variance of X_{1k}, a centered random variable, is given by[18]

$$\sigma_{x1k}^2 = E\left\{\left[\sum_{j=1}^{k-1} (\lambda_1^{k-j-1}\beta_{1j} + \ldots + \lambda_j^{k-j-1}\beta_{ij})\right]^2\right\}$$

$$= \sum_{j=1}^{k-1} \left\{E[(\lambda_1^{k-j-1}\beta_{1j})^2] + \ldots + E[(\lambda_j^{k-j-1}\beta_{ij})^2]\right\}$$

$$= \sum_{j=1}^{k-1} \left\{(\lambda_1^2)^{k-j-1} E[(\beta_{1j})^2] + \ldots + (\lambda_j^2)^{k-j-1} E[(\beta_{ij})^2]\right\} \tag{49}$$

Since the variance of each of the quantization errors is known, the variance of each $\beta_{\ell j}$ can be determined from equation (48). This quantity is independent of j. Hence equation (49) can be rewritten

$$\sigma^2_{x1k} = \sigma^2_{\beta 11} \sum_{j=1}^{k-1} (\lambda_1^2)^{k-j-1} + \ldots + \sigma^2_{\beta j1} \sum_{j=1}^{k-1} (\lambda_j^2)^{k-j-1} \qquad (50)$$

This equation gives the mean square tracking error at the sampling instant in terms of quantities which can be determined from system parameters and the parameter h which describes the quantization process. A similar computation could be used to determine the variance of any component of $\underline{X}_k$ or $\underline{m}_k$.

The main conclusion to be reached from equation (50) is that if the eigenvalues of Λ have magnitudes less than one, then the mean square error at the $k^{\underline{th}}$ sampling instant approaches a limit as k increases, this limit being a function of system parameters and h. Examples of this computation are included in the next chapter.

Changes in Sampling Period Length

A large class of possible applications of the synthesis technique which has been presented would involve a time-shared digital computer. The computer would be shared between several systems. In such situations it is likely that the number of loops in operation would vary with time, and, therefore, the sampling period for each loop would vary. The more tracking systems in operation the longer it would take the computer to perform computations for them all, and each system would effectively see a longer sampling period. The effect of variations in sampling period length should, therefore, be investigated as part of the design process.

The method which has been employed in the example problems to judge system performance sensitivity to changes in sampling period length is to evaluate system frequency response via equation (41) for several values of T (the sampling period length). This can be done easily with the aid of a digital computer. The work involved is essentially negligible in comparison to that involved in determination of system parameters - i.e. the control algorithm. In the sample problems worked it has been found that system frequency response deteriorates, in general, as the sampling period length is changed; either shortened or lengthened.

Another method of judging system performance sensitivity to changes in sampling period length is to define

$$\underline{S}_T(k) \equiv \frac{\partial \underline{y}_k}{\partial T} = \lim_{\Delta T \to 0} \frac{\underline{y}[k(T+\Delta T)] - \underline{y}(kT)}{\Delta T} \qquad (51)$$

to be the sensitivity of $\underline{y}_k$ to the sampling period length T. Here $\underline{y}_k$ is the vector which has been defined in Chapter IV.

$$\underline{y}_k = \begin{bmatrix} \underline{x}_k \\ \\ \underline{m}_k \end{bmatrix}$$

Now it is known that

$$\underline{y}_k = \underline{y}(kT) = \Lambda\, \underline{y}[(k-1)T] + \Omega\, \underline{r}[(k-1)T] \qquad (52)$$

where Λ and Ω are defined immediately after equation (38). Equation (52) can be differentiated with respect to T with the following result.

$$\frac{\partial \underline{y}_k}{\partial T} = \Lambda \frac{\partial \underline{y}_{k-1}}{\partial T} + \Omega \frac{\partial \underline{r}_{k-1}}{\partial T} + \frac{\partial \Lambda}{\partial T} \underline{y}_{k-1} + \frac{\partial \Omega}{\partial T} \underline{r}_{k-1} \tag{53}$$

Since Ω does not depend on T (see equation (38)) the last term of equation (53) may be dropped. Only the elements of Λ which are associated with the plant depend on T since the control element is fixed. The elements of Λ which are functions of T will, in many cases, have small derivatives with respect to T. In such cases equation (53) reduces to

$$\frac{\partial \underline{y}_k}{\partial T} = \Lambda \frac{\partial \underline{y}_{k-1}}{\partial T} + \Omega \frac{\partial \underline{r}_{k-1}}{\partial T} \tag{54}$$

or, in terms of the sensitivity measure

$$\underline{S}_T(k) = \Lambda \underline{S}_T(k-1) + \Omega \frac{\partial \underline{r}_{k-1}}{\partial T} \tag{55}$$

which is a linear difference equation whose solution determines the sensitivity of system performance to changes in sampling period length. This equation will be solved in sample cases in the next chapter. It should be emphasized that the structure of this equation is the same as that of equation (13) which describes the overall system. Thus a good understanding of the system itself implies a good understanding of system performance sensitivity to changes in sampling period length.

System Stability

It is possible to formulate several useful definitions of system stability. The definition which will be used here is the following: "A hybrid tracking system is said to be stable if the transient solution of the difference equation describing the system indicates that the system approaches an equilibrium state after any disturbance."

The transient solution of the difference equation (38) governing the hybrid tracking system of interest here is

$$\underline{y}_k = \Lambda^k \ \underline{y}_o \tag{56}$$

Application of Sylvester's theorem indicates that

$$\underline{y}_k = \sum_{j=1}^{j} \lambda_j^k \Lambda_j$$

where

$$\Lambda_j = \lim_{\lambda \to \lambda_j} \frac{f(\lambda)(\lambda I - \Lambda)^{-1}}{f'(\lambda)}$$

and $\lambda_1, \ldots, \lambda_j$ are the eigenvalues of Λ.

Thus a sufficient condition for system stability is that

$$|\lambda_j| < 1 \qquad j \leq j$$

Once the system has been synthesized, Λ is known. In fact, Λ is known after the first minimization. At this point the sy stem can be tested for stability. It is possible that an unstable system could be altered slightly, thereby made stable, and still yield acceptable tracking performance. This possibility would have to be investigated in individual cases. No general statement is possible.

CHAPTER VI

EXAMPLES

This chapter presents the results of application of the procedure to some specific problems. In particular, two different plants are considered, and, for one of these, two different performance indices are considered for the first minimization for comparison purposes. All of these examples are not needed to illustrate the details of application of the procedure. Therefore, one example is considered in detail and only features of the other examples which supplement the first example are presented.

Example One

In this example the control element will be synthesized for use with a plant whose dynamic behavior is described by the following differential equation.

$$\ddot{X} + 2\dot{X} + 2X = m(t)$$

Letting $X = X_1$, $\dot{X}_1 = X_2$, this equation can be converted to the following vector form.

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} m(t)$$

Assuming m(t) constant between sampling instants the solution of this equation is given by

$$\begin{bmatrix} X_1 \\ \\ X_2 \end{bmatrix} = e^{-(t-t_o)} \begin{bmatrix} \cos(t-t_2) + \sin(t-t_o) & \sin(t-t_o) \\ \\ -2\sin(t-t_o) & \cos(t-t_o) - \sin(t-t_o) \end{bmatrix} \begin{bmatrix} X_1(t_o) \\ \\ X_2(t_o) \end{bmatrix}$$

$$+ \int_{t_o}^{t} \begin{bmatrix} \sin(t-\alpha-t_o) \\ \\ \cos(t-\alpha-t_o) - \sin(t-\alpha-t_o) \end{bmatrix} e^{-(t-\alpha-t_o)} d\alpha$$

To obtain the difference equation description of the plant, set

$$t_o = t_k$$

$$t_o = t_{k+1}$$

where

$$t_{k+1} - t_k = T$$

The difference equation is found to be

$$\begin{bmatrix} X_1(kT) \\ \\ X_2(kT) \end{bmatrix} = \begin{bmatrix} 0.9976 & 0.0475 \\ \\ -0.0951 & 0.9025 \end{bmatrix} \begin{bmatrix} X_1[(k-1)T] \\ \\ X_2[(k-1)T] \end{bmatrix} + \begin{bmatrix} 0.00121 \\ \\ 0.0475 \end{bmatrix} m_{k-1}$$

where a value of $T = 0.05$ sec. has been used. This part of the synthesis procedure was developed in general terms in Chapter III and the above difference equation corresponds to equation (5) of that chapter. This difference equation description is necessary for implementation of the first minimization as discussed in Chapter IV.

It is assumed in this example that the power spectrum of the reference signal is flat from zero to ten radians per second and that the reference signal contains no energy at frequencies higher than ten radians per second.

Case I

Assuming a sinusoidal reference signal of frequency ω, the following difference equation can be written to describe the change in both plant state and reference signal state between the k^{th} and $k + 1^{st}$ sampling instants.

$$\begin{bmatrix} X_{1k+1} \\ X_{2k+1} \\ r_{1k+1} \\ r_{2k+1} \end{bmatrix} = \begin{bmatrix} 0.9976 & 0.0475 & 0 & 0 \\ -0.0951 & 0.9025 & 0 & 0 \\ 0 & 0 & \cos \omega T & \frac{1}{\omega} \sin \omega T \\ 0 & 0 & -\omega \sin \omega T & \cos \omega T \end{bmatrix} \begin{bmatrix} X_{1k} \\ X_{2k} \\ r_{1k} \\ r_{2k} \end{bmatrix} + \begin{bmatrix} 0.00121 \\ 0.0475 \\ 0 \\ 0 \end{bmatrix} m_k \qquad (57)$$

Defining the vector $\underline{Z}_k$ to be

$$\underline{Z}_k = \begin{bmatrix} X_{1k} \\ X_{2k} \\ r_{1k} \\ r_{2k} \end{bmatrix}$$

a meaningful index of system performance is

$$S = \sum_{k=o}^{\infty} \underline{Z}_k{}' Q \, \underline{Z}_k \qquad (58)$$

where the prime indicates transpose and

$$Q = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

The quantity S is seen to be the sum of squared tracking error plus tracking error rate at all sampling instants. This performance index was discussed in general terms in Chapter IV and corresponds to equation (16) of that chapter with the parameter $\lambda = 1$.

The value of the performance index S is a functional of the sequence $\{m_k\}$, and the minimizing sequence may be found using dynamic programming. The validity of assuming m_k to be of the form

$$m_k = \varphi_d m_{k-1} + \Delta_D \underline{Z}_{k-1}$$

has been argued in Chapter IV. Repeated substitution of this and equation (57) into equation (58) shows that S, and, in particular, the minimum value of S, can be expressed in the form

$$S_{j\,min} = \sum_{k=j}^{\infty} \underline{Z}_k{}' Q \, \underline{Z}_k = \underline{Z}_k{}' \mathcal{Q} \, \underline{Z}_k + m_k \mathcal{H} m_k + m_j \mathcal{K} \underline{Z}_j + \underline{Z}_j{}' \mathcal{K}' m_j \qquad (59)$$

where $\mathcal{Q}$, $\mathcal{H}$, $\mathcal{K}$ are symmetric matrices.

The principle of optimality applied to this situation states that

$$S_{j\,min} = \min_{m_j} \left\{ \underline{Z}_j{}' Q \, \underline{Z}_j + S_{j+1\,min} \right\} \qquad (60)$$

Substituting equations (57) and (59) into this expression and performing the minimization operation shows that

$$m_j = -\mathcal{H}^{-1} \mathcal{K} Z_j$$

In order to specify the algorithm for generating the control sequence the matrices $\mathcal{Q}$, $\mathcal{H}$, $\mathcal{K}$ must be determined. This is done by substituting equations (57), (59), and (61) into equation (60) to obtain an identity. The solution process is tedious and has been presented in Chapter IV. The result is that equation (61) becomes

$$m_j = -0.9489\, m_{j-1} - 16.62X_{1k-1} - 18.47X_{2j-1} + 20.30r_{1j-1} + 20.52r_{2j-1}$$

It will be remembered that this development has assumed a one radian per second sinusoidal reference signal. Since the reference signal contains energy over a band of frequencies from zero to ten radians per second, the minimization is repeated for $\omega = 2, 3, \ldots, 10$. It has been shown in Chapter IV that only the input gains (those associated with $\underline{r}_{j-1}$) will change with frequency.

The optimum input gains for use with the various frequencies contained in the reference signal being known from repeated execution of the minimization, a fixed set of gains must be selected for use with all of these frequencies. A quantitative comparison of the fixed gain system with the optimum systems is established by equation (36). It is found that gains of 20.37 and 20.56 for r_k and $\dot{r}_k$ respectively result in minimum mean square deviation of the fixed system frequency characteristic (gain magnitude) from that of the optimum over the frequency band containing the reference signal.

The overall synthesized system is described by

$$\begin{bmatrix} X_{1k+1} \\ X_{2k+1} \\ m_{k+1} \end{bmatrix} = \begin{bmatrix} 0.9976 & 0.0475 & 0.0012 \\ -0.0951 & 0.9025 & 0.0475 \\ -16.62 & -18.47 & -0.9489 \end{bmatrix} \begin{bmatrix} X_{1k} \\ X_{2k} \\ m_k \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 20.37 & 20.56 \end{bmatrix} \begin{bmatrix} r_{1k} \\ r_{2k} \end{bmatrix}$$

where $X_{1k} = X(kT)$, $X_{2k} = \dot{X}(kT)$, etc. This equation corresponds to equation (13).

In selecting input gains no importance has been attached to the phase shift characteristic of the input structure. It will be seen in the plots which follow that the phase shift characteristics of the synthesized system are completely satisfactory in that the phase shift from input to output is proportional to the frequency of the input signal.

The characteristic polynomial of the 3 x 3 transition matrix of the synthesized system is

$$\mu^3 + (-0.9489)\mu^2 + (-0.0035)\mu + 0.0016 = 0$$

which has the following roots.

$$\mu_1 = 0.952$$

$$\mu_2 = 0.039$$

$$\mu_3 = -0.042$$

These are the eigenvalues of the transition matrix and, since they are all less than one in absolute value, the synthesized system is stable as was shown by application of Sylvester's theorem in Chapter V.

The response of the system to the reference signal

$$r(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

is plotted in Figure 4. The corresponding control signal is plotted in Figure 5. One common reason for including a plant input energy term in the performance index is to insure that extremely large plant input signals do not occur. The plot of the control signal is included to show that the synthesis procedure does not cause extremely large plant input signals. The response of the synthesized system to the reference signal

$$r(t) = U_{-2}(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases}$$

is plotted in Figure 6. In both of these plots the plant vector has the value

$$\begin{bmatrix} X(0) \\ \dot{X}(0) \end{bmatrix} = \begin{bmatrix} X_{10} \\ X_{20} \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

at t = 0 when the system begins to function. The frequency response of the system is plotted in Figure 7. In Figure 8 the frequency response of the system is plotted for several sampling period lengths which differ from the sampling period length for which the system was designed. These plots show that system frequency response is not strongly affected by changes in sampling period length.

A further measure of system performance sensitivity to changes in sampling period length is obtained by solving equation (55). For the case of a step input, the solution is

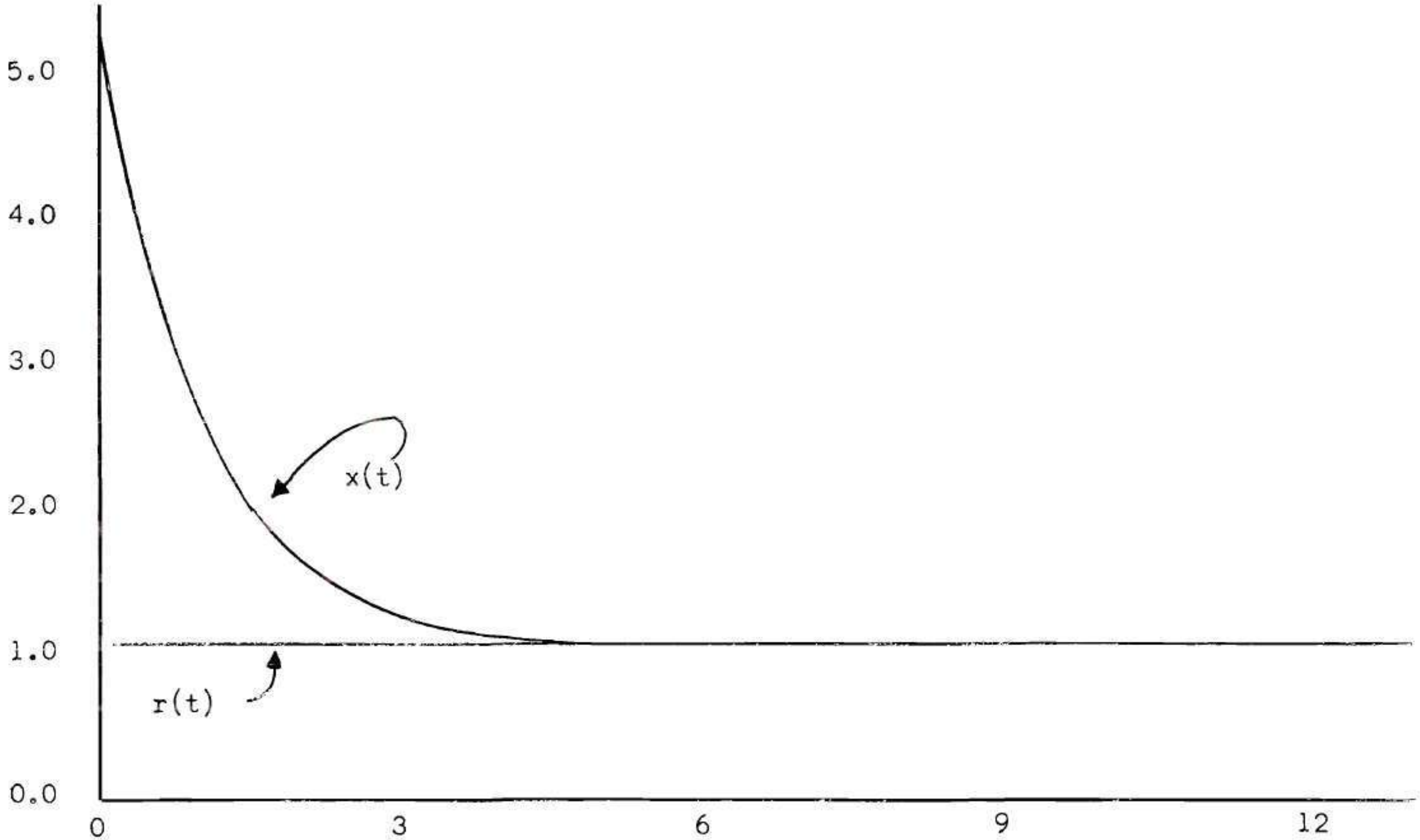


Figure 5. System Step Response, Example 1 - Case I.

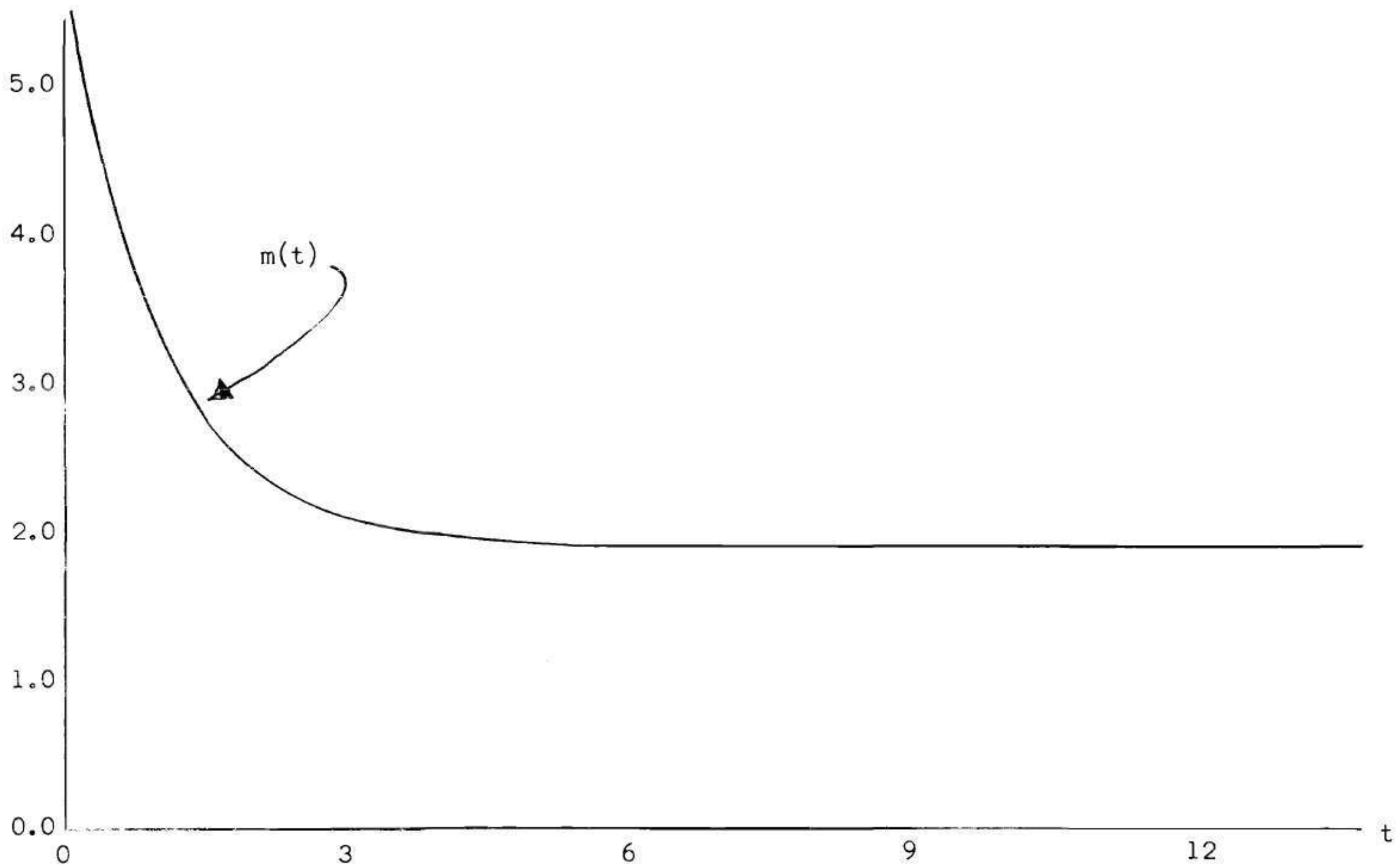


Figure 6. Control Signal for Step Input, Example 1 - Case I.

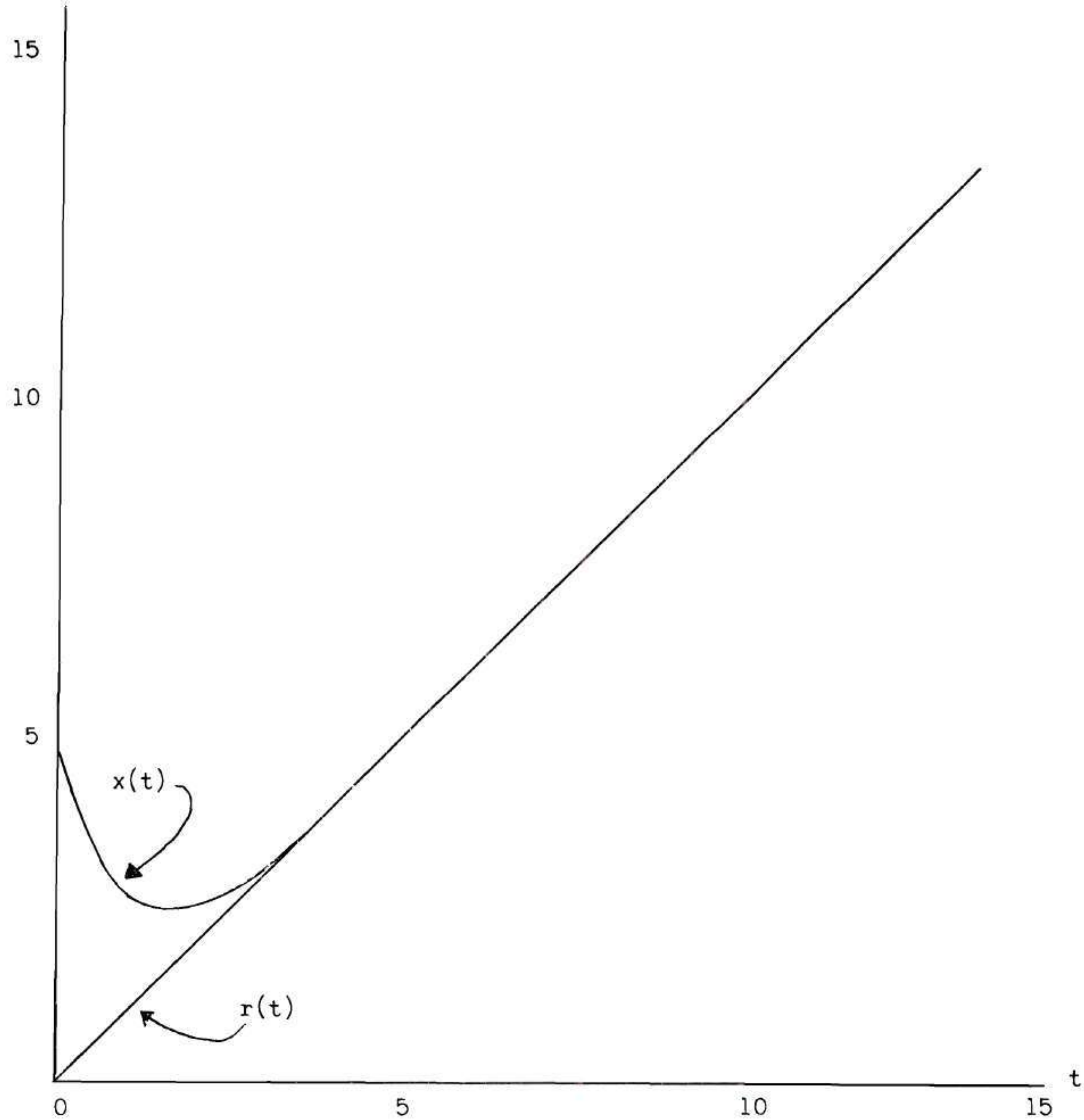


Figure 7. System Ramp Response, Example 1 - Case I.

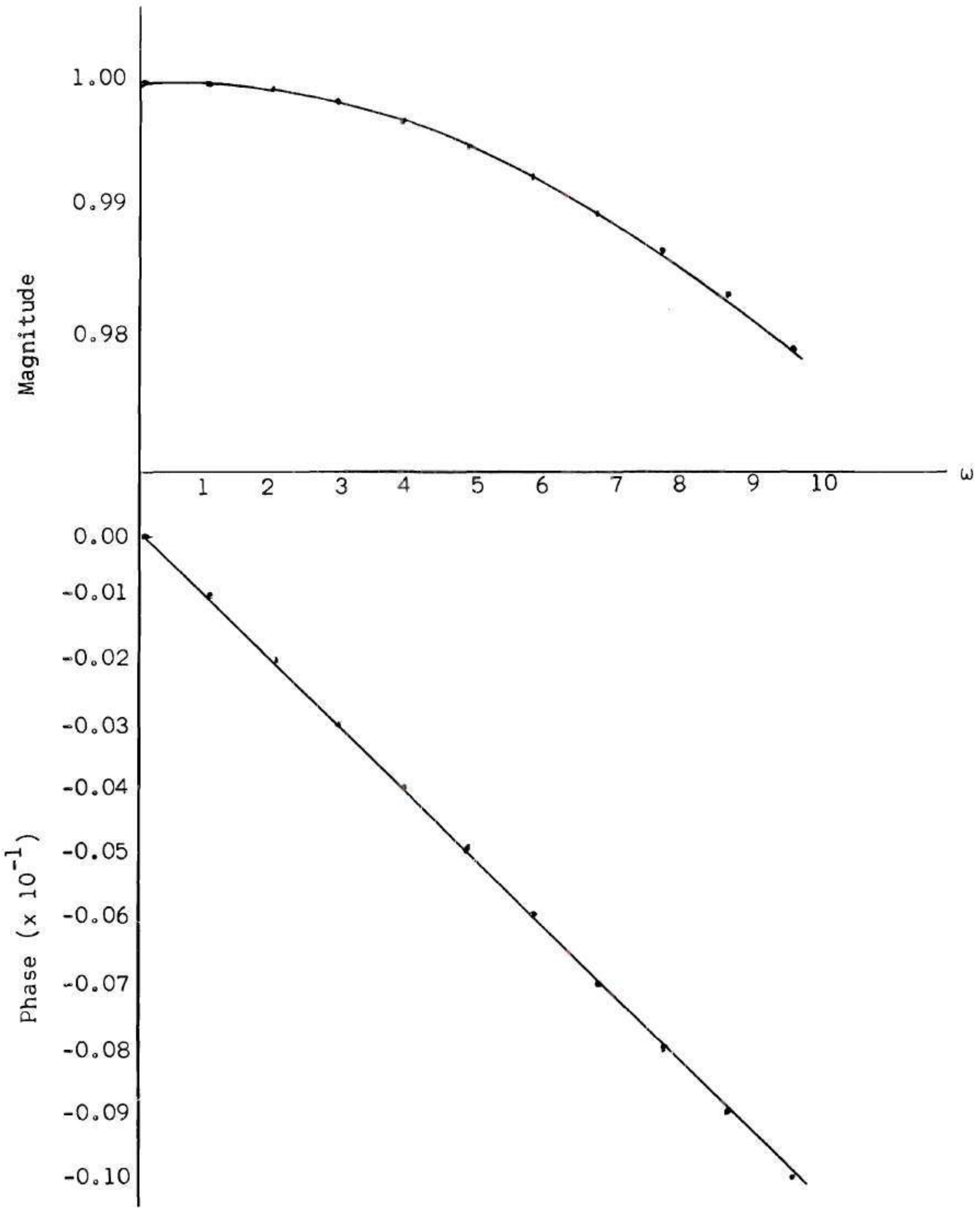


Figure 8. System Frequency Response with 0.05 Second Sampling Period, Example 1 - Case I.

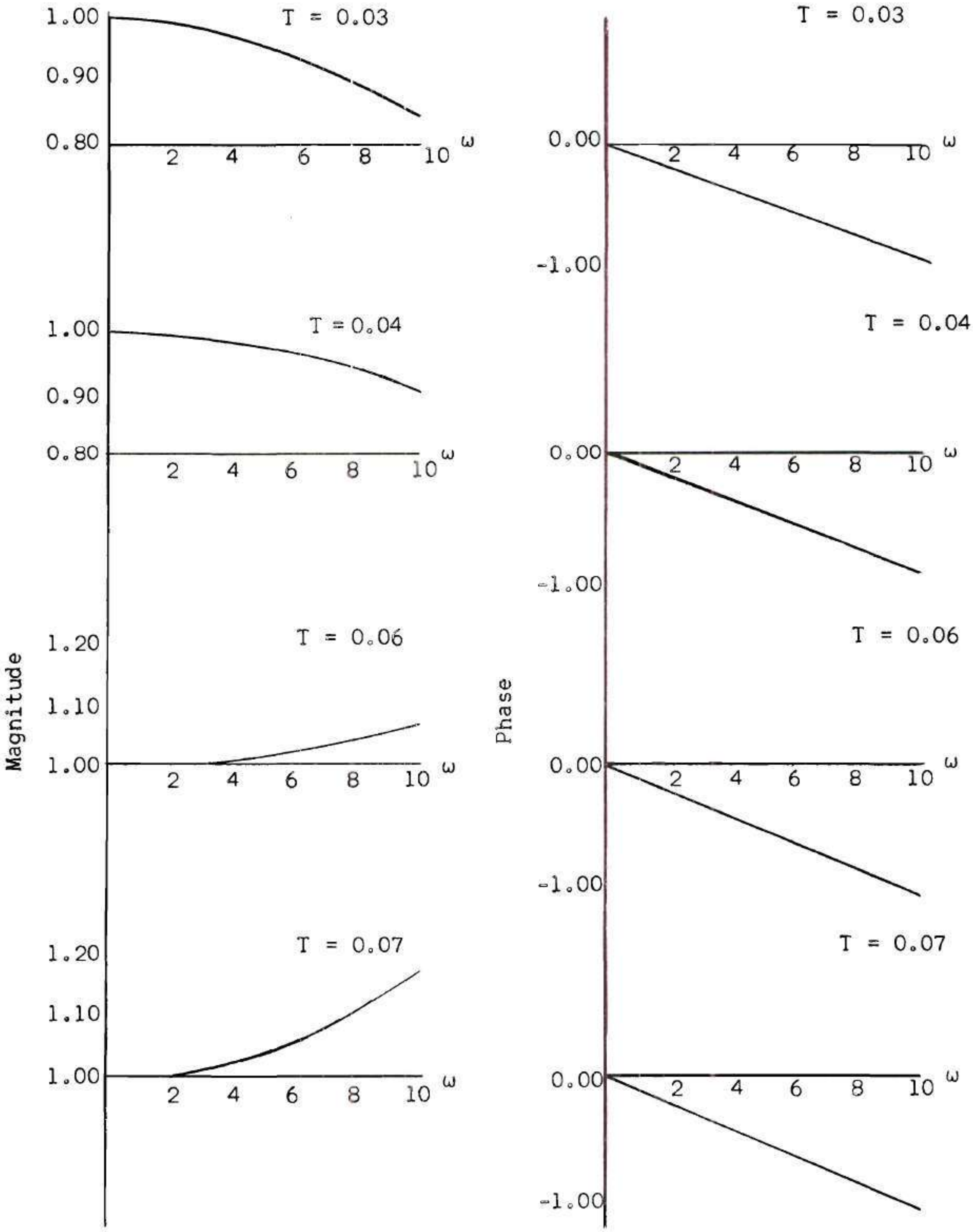


Figure 9. System Frequency Response with Several Sampling Period Lengths, Example 1 - Case I.

$$\begin{bmatrix} \frac{\partial X_{1k}}{\partial T} \\ \frac{\partial X_{2k}}{\partial T} \\ \frac{\partial m_k}{\partial T} \end{bmatrix} = \begin{bmatrix} 0.9976 & 0.0475 & 0.0012 \\ -0.0951 & 0.9025 & 0.0475 \\ -16.62 & -18.47 & -0.9489 \end{bmatrix}^k \begin{bmatrix} \frac{\partial X_{10}}{\partial T} \\ \frac{\partial X_{20}}{\partial T} \\ \frac{\partial m_o}{\partial T} \end{bmatrix}$$

By Sylvester's theorem, this solution approaches zero (the null vector) as k approaches infinity. For the case of a unit ramp input, the solution of the sensitivity equation (55) would be the same as the system response to a unit step, which has already been plotted in Figure 4. This is true because for a unit ramp,

$$\frac{\partial \underline{r}_k}{\partial T} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad k > 0$$

Thus a change in sampling period length leads to a steady state error in the ramp response, the error being numerically equal to the slope of the ramp times the change in sampling period length. This expression is valid only for small changes in sampling period length on which basis equation (55) was derived.

Since the control element is to be of a digital nature, quantization will be present. The variables fed into and out of the control element will be quantized. Thus the control signal will be incorrect and this error will propagate through the plant and appear as tracking error. Chapter V contains an analysis of the effect of quantization, and application of that development to the present situation shows that the maximum tracking error which can arise due to quantization is given by

$$|\text{Error in } X_{1k}| \leq 4.413\ h$$

where h is the spacing between quantization levels and the number 4.413 is related to plant and control element parameters. The mean square value of the tracking error due to quantization is found to be

$$\sigma^2 = 41.91\ \frac{h^2}{3}$$

again by application of the development of Chapter V. Here h is the same as above and the number 41.91 again depends on (can be determined from) plant and control element parameters.

In summary, a control element has been determined for use with a given plant such that the overall system is a tracking system. That is, the plant output is a reproduction of the reference signal and although a flat power spectrum has been assumed here any shape could have been handled with equal ease. The parameters of the system are different from those which would be determined by classical techniques indicating that the method which has been developed is more versatile. That is, the error signal is not just the difference between output and input. Errors arising from the digital nature of the control element have been investigated and equations have been found which relate tracking error to quantization error. This enables the designer to determine the necessary quantization fineness in terms of permissible tracking error. System performance sensitivity to changes in sampling length has also been investigated. This sensitivity must be accepted because there is no way to minimize it during the design procedure. Knowing the sensitivity and the permissible tracking error permits the designer to specify allowable changes in sampling period length.

Case II

The only thing which is changed in this case from the previous case is that the performance index for the first minimization is changed. The performance index used in this case is that of equation (16) with $\lambda = 0.50$. The details of the synthesis are the same as before and are omitted. The resulting synthesized system is described by the following difference equation

$$\begin{bmatrix} x_{1k+1} \\ x_{2k+1} \\ m_{k+1} \end{bmatrix} = \begin{bmatrix} 0.9976 & 0.0475 & 0.00121 \\ -0.0951 & 0.9025 & 0.0475 \\ -24.79 & -19.04 & -0.9684 \end{bmatrix} \begin{bmatrix} x_{1k} \\ x_{2k} \\ m_k \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 28.73 & 20.31 \end{bmatrix} \begin{bmatrix} r_{1k} \\ r_{2k} \end{bmatrix}$$

The step response, control signal for the step response, and the ramp response are shown in Figures 9, 10, and 11 respectively. These are presented so that comparison can be made with the corresponding data of the previous case in order to see what effect the performance index has on system behavior. The effect of decreasing the penalty for derivative error leads to higher gains and faster response. The effect of further decreasing the penalty will lead to an unstable system with $\lambda = 0$ in the performance index for the first minimization. With $\lambda = 0$ the feedback and input gains turn out to be close to 800 as compared with 20 in the above equation. This aspect of system design is discussed further in the next example.

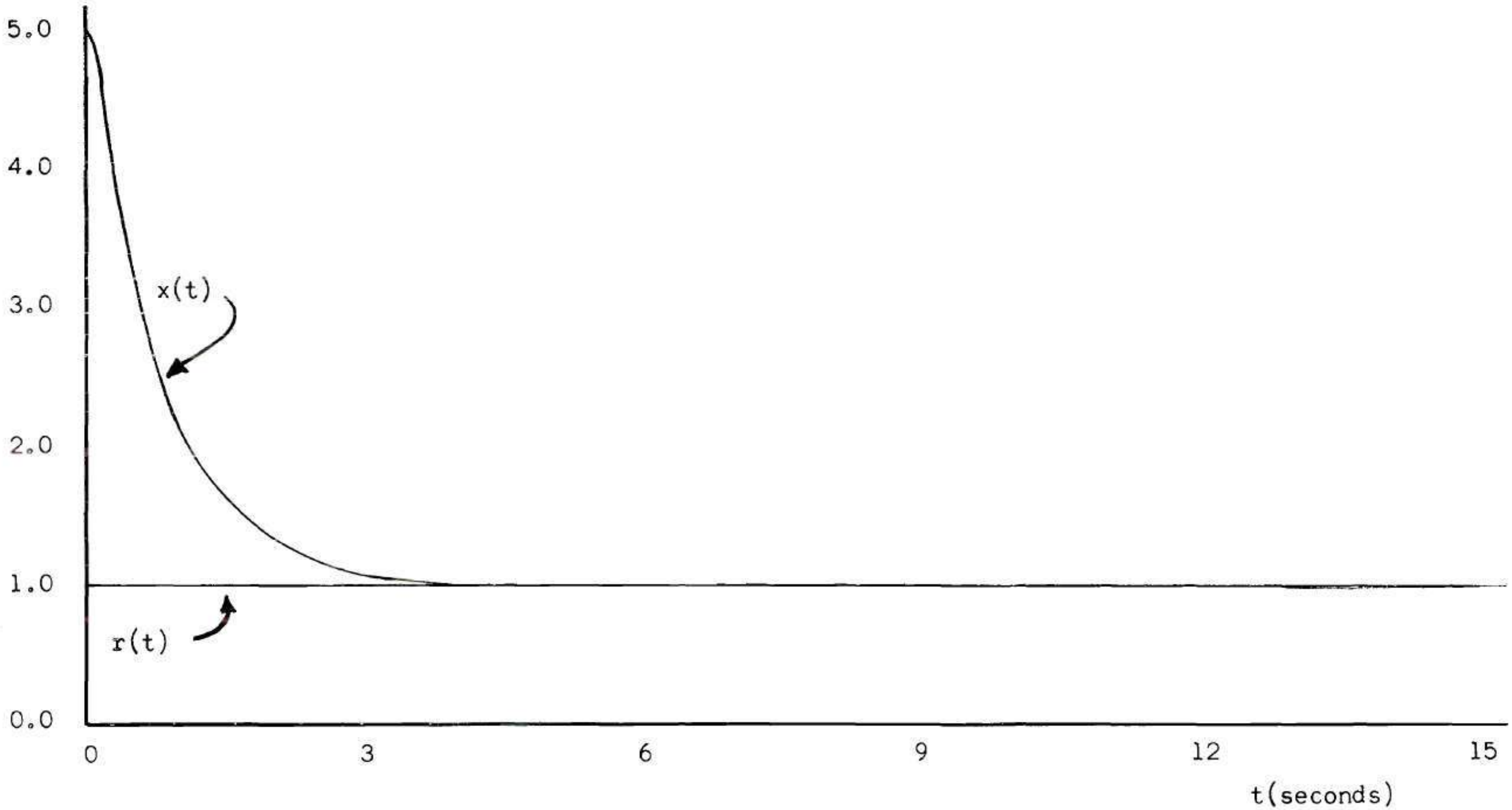


Figure 10. System Step Response, Example 1 - Case II.

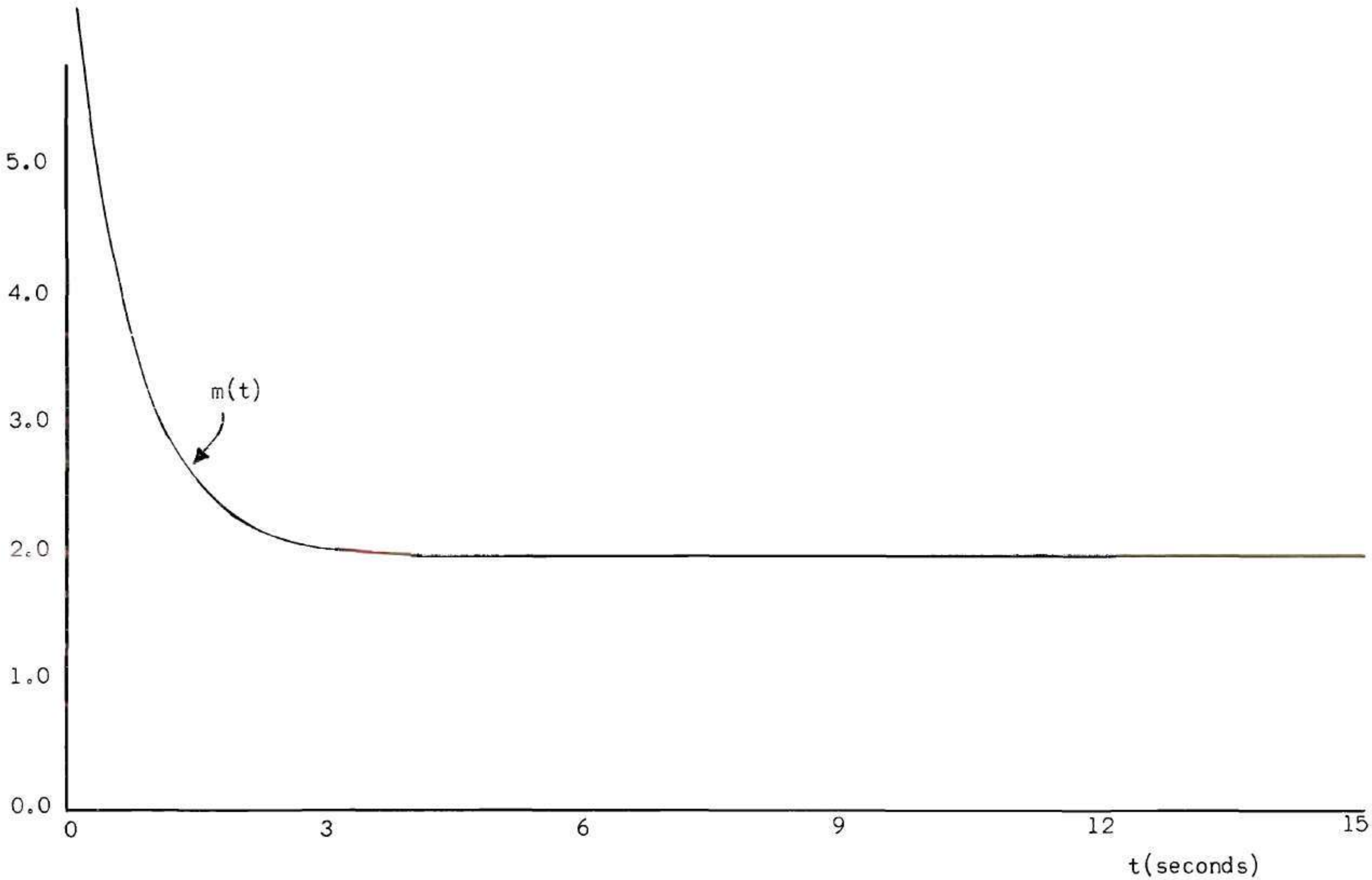


Figure 11. Control Signal for Step Input, Example 1 - Case II.

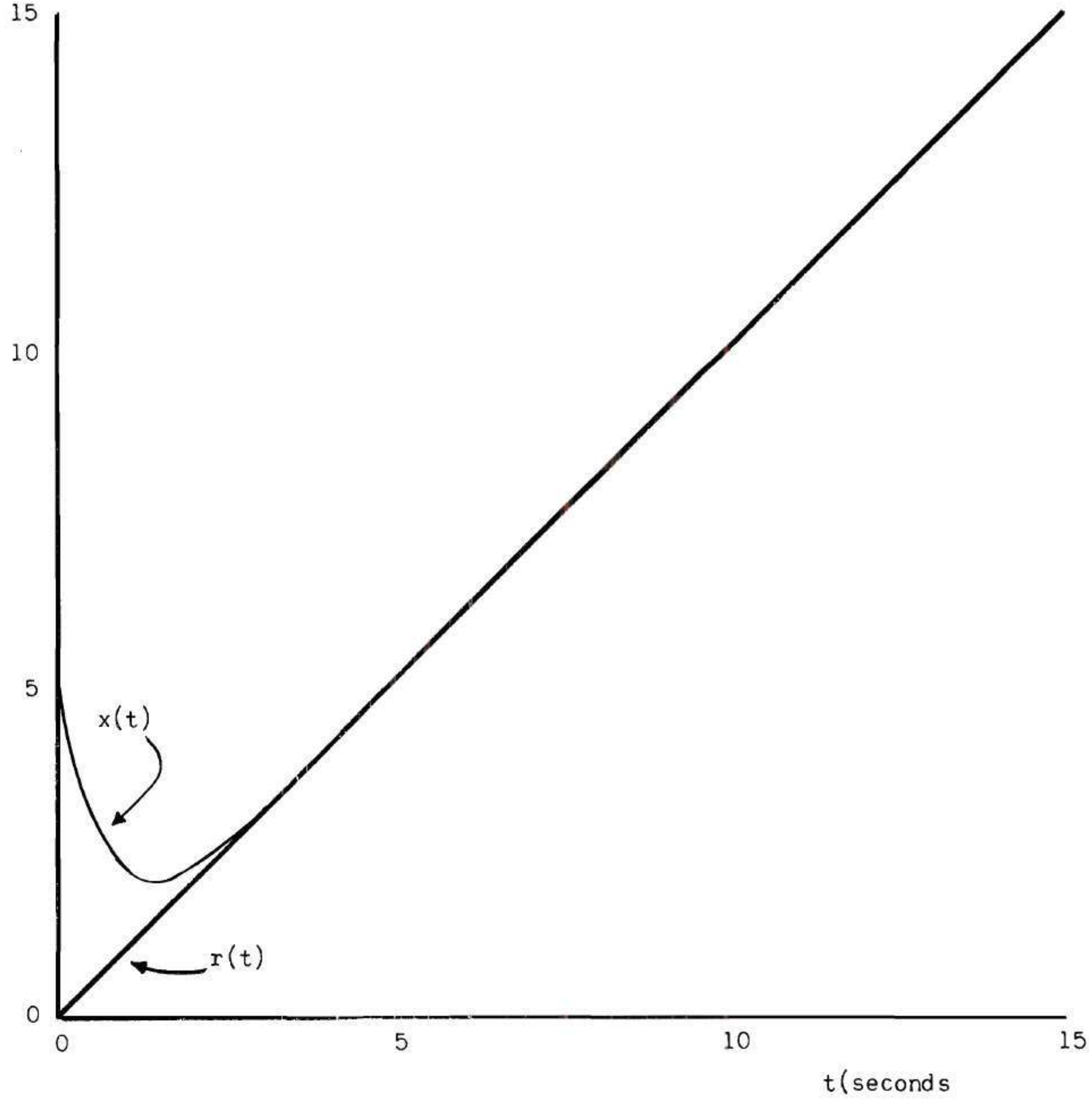


Figure 12. System Ramp Response, Example 1 - Case II.

Example Two

The purpose of this example is to indicate that although the design method which has been developed is a synthesis method, there is still some trial and error involved. The point is that even though a system can be determined which will minimize the performance index, selection of the performance index involves, in general, some trial and error. In the preceding example two different performance indices for the first minimization were considered. It can be seen from the plots for that example that the lower value of $\lambda = 0.50$ produces a faster response. In this example the first minimization is carried out with the parameter $\lambda = 1.00$ and then with $\lambda = 0.00$ and differences in the resulting systems are noted.

In this example the plant is described by the following differential equation.

$$\ddot{X} + 2\dot{X} + 17X = m$$

The synthesis is carried out on the basis of a 0.05 sec. sampling period length and a reference signal having a uniform distribution of energy from zero through ten radians per second. Using the performance index indicated in equation (16) with $\lambda = 1.00$ for the first minimization and then performing the second minimization leads to the following system.

$$\begin{bmatrix} X_{1k+1} \\ X_{2k+1} \\ m_{k+1} \end{bmatrix} = \begin{bmatrix} 0.9795 & 0.0472 & 0.0205 \\ -0.7932 & 0.8850 & 0.0476 \\ 11.88 & -17.20 & -0.9135 \end{bmatrix} \begin{bmatrix} X_{1k} \\ X_{2k} \\ m_k \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 20.70 & 20.64 \end{bmatrix} \begin{bmatrix} r_{1k} \\ r_{2k} \end{bmatrix}$$

The response of this system does not differ significantly from the response previously plotted, and therefore only the step response and the frequency response are plotted. These plots are presented in Figures 12 and 13 respectively.

The system determined above is stable. However, using the same design data except setting $\lambda = 0.00$ in the performance index for the first minimization leads to an unstable system. It would be expected from consideration of the continuous case that as λ approaches zero, system gains would increase. In the continuous case for a system with the plant under consideration here, no matter how large the gains are made, the system will be stable. However, because of the time lag associated with the digital portion of the system, high gains in the present example result in instability. This is not too surprising since the method is basically a steady state design method.

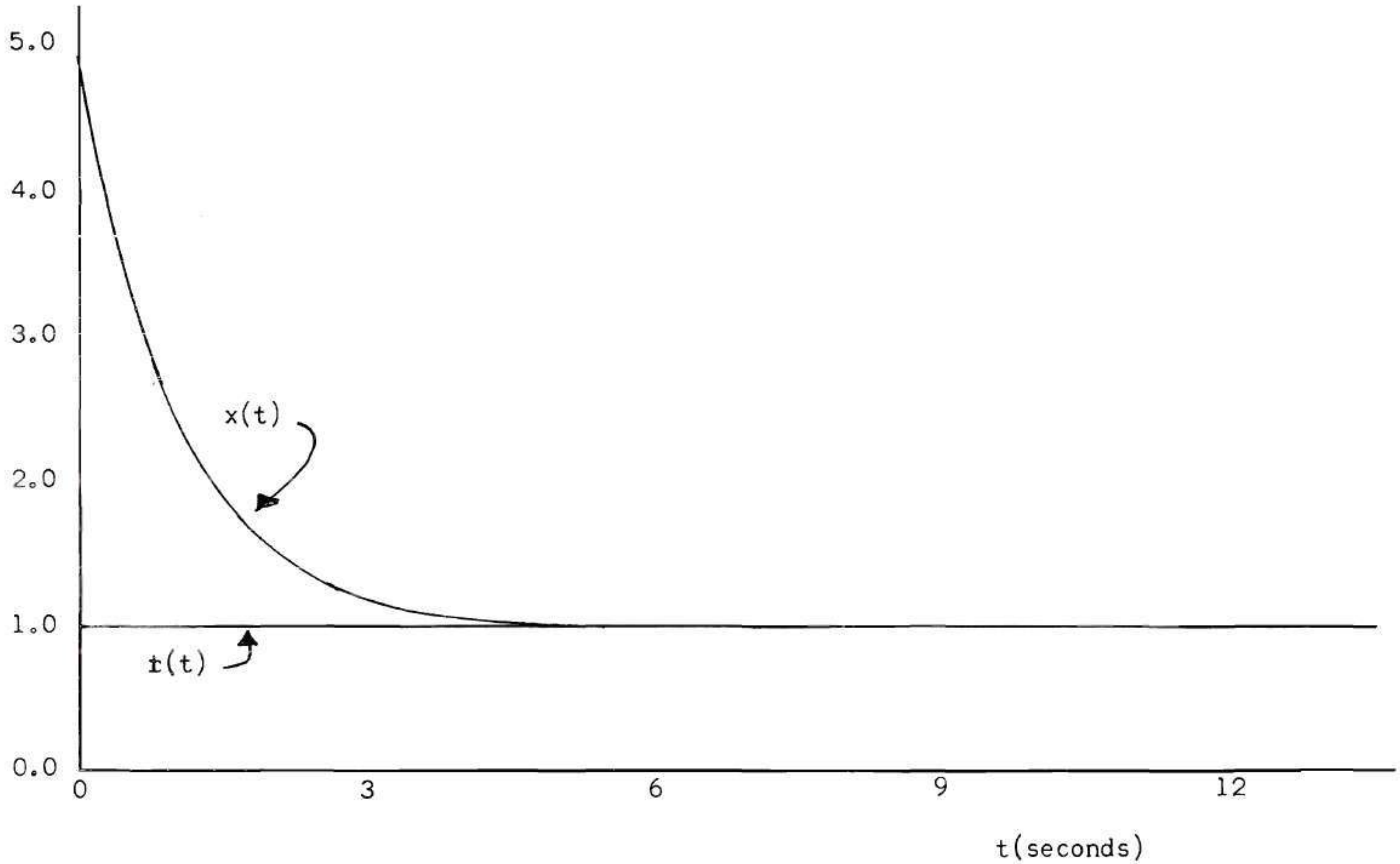


Figure 12. System Step Response, Example 2.

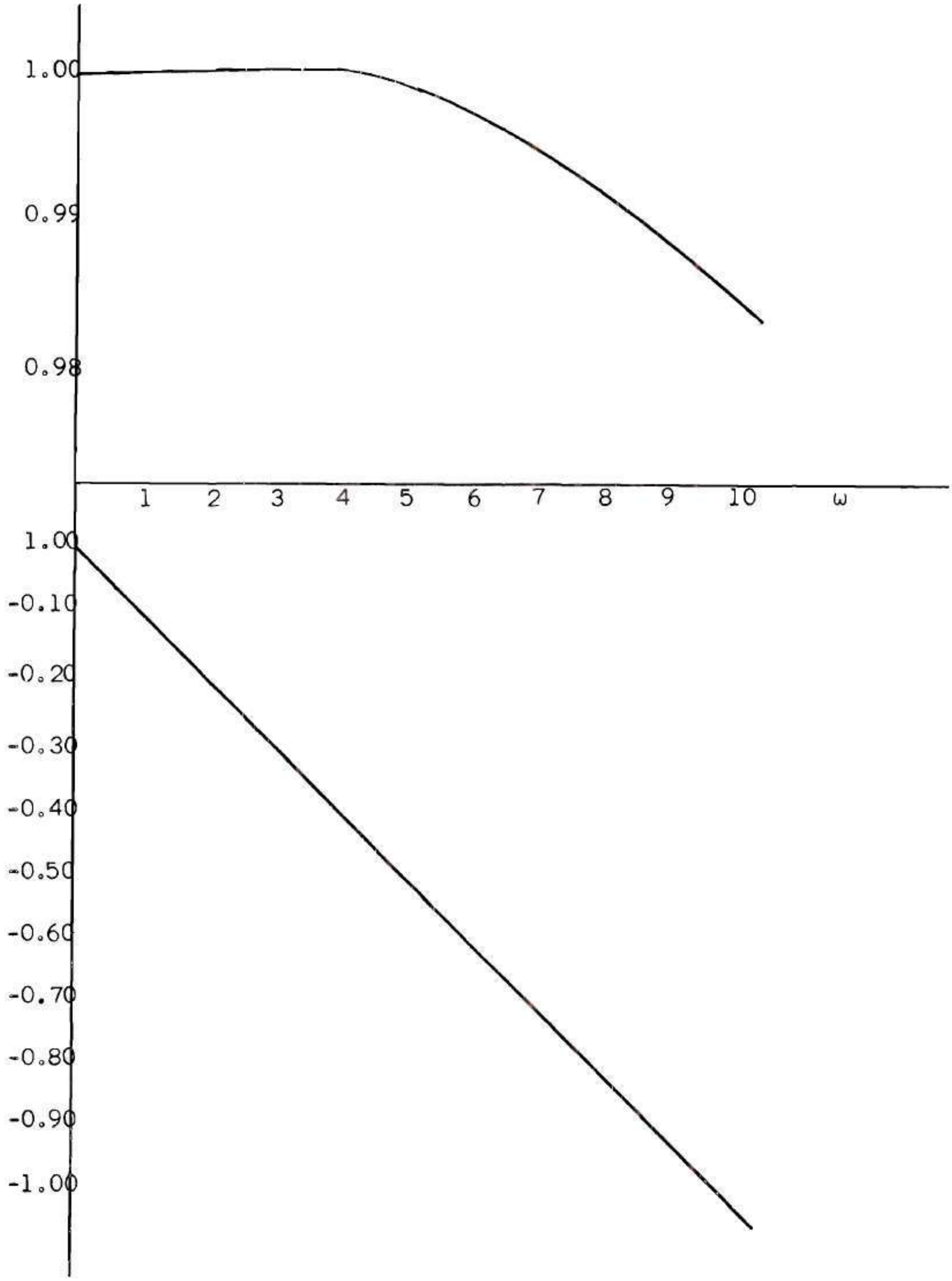


Figure 14. System Frequency Response with 0.05 Second Sampling Period, Example 2.

CHAPTER VII

CONCLUSIONS

In the preceding pages a method has been developed for synthesis of a class of hybrid tracking systems. Modern methods of system synthesis were exploited to develop a method for determining the structure and parameters of the system for minimum tracking error with a sinusoidal reference signal. This method was developed for an arbitrary reference signal frequency. For the case of a reference signal which admits a Fourier series representation this method can be used to find the optimum tracking system for each frequency present in the series. It has been shown that the optimum system is a feedback system and that the feedback structure and parameters are the same no matter what the reference signal frequency may be. Tracking system synthesis is completed by selecting input gains or parameters. For the optimum system, these gains are different for each different reference signal frequency. A method was developed for selecting a set of fixed input gains to be used with all frequencies present in the reference signal from the previously determined optimum input gains for the various frequencies present in the actual reference signal. The modern methods of system synthesis which have been used require that a feasible system performance index be written. This can be done only for a limited class of reference signals, including sinusoids. The idea of synthesizing a tracking system for several reference signals for which a suitable performance index can be written and then using superposition to find a system for use with the sum of these reference signals, for which a suitable performance index cannot be written, forms the innovation which is the basis of this thesis. The same idea could be applied using orthogonal functions other than sinusoids. The requirements on such a set of functions would be that they satisfy a known differential equation and that a useful class of reference signals admit expansion as series of the function.

Practical implementation of the method is possible because, for the optimum system, only input gains vary with the frequency of the reference signal. If all of the optimum system gains, the feedback gains as well as the input gains, varied with reference signal frequency, the process of selecting a fixed set of gains for use with a reference signal containing many frequencies would be extremely tedious and there would be no way of knowing that a system so determined would provide good tracking performance. Since only input gains depend on reference signal frequency, the input gains can be visualized as a filter and, through frequency response, the effect of variations in the input gains can be determined. In this process the well developed frequency domain intuition which most system designers possess can be brought to bear.

The procedure can be applied in cases where the reference signal has a continuous spectrum. There is no essential difference between this and the case of a reference signal containing only discrete frequencies. The reference signal is modeled by a set of discrete frequencies and in the performance index used for selecting input gains the deviation of the actual system from the optimum system is weighted at each frequency according to the relative amount of energy in the reference signal near that frequency.

In the process of selecting the set of fixed input gains to be used in the actual system it is possible to decrease system sensitivity to noise. This involves adding to the performance index used for selecting input gains terms which depend on the magnitude of the input gain at frequencies containing noise energy but no reference signal energy. At these frequencies the input gain of the system should be zero so that no response to noise signals is possible. In using this method of decreasing system sensitivity to noise any filtering characteristics of the plant or other systems which supply the reference signal should be noted and taken advantage of so that system gains are determined on the basis of tracking performance in so far as possible.

Sensitivity analyses have been developed which allow quantitative determination of system performance sensitivity to quantization and changes in sampling period length. This can be done as part of the design routine and is an important feature of the method since in any practical application such sensitivity measures would be necessary. System performance sensitivity to quantization is measured by a bound on tracking error due to quantization and the mean square tracking error due to quantization. System performance sensitivity to changes in sampling period length has been found in example problems to be adequately measured by consideration of the changes produced in system frequency response by changes in sampling period length. An additional method of judging system performance sensitivity to changes in sampling period length has been included which gives the sensitivity as a function of system parameters and the reference signal. The results of the sensitivity analyses would have to be considered in the light of individual applications and the corresponding requirements. No general statement is possible as to whether the method will yield a sufficiently insensitive system.

Selection of input gains is based on knowledge of the distribution of reference signal energy with frequency. That is, the input gains depend on the power spectrum of the reference signal. It is possible to determine system input gains for several possible reference signal spectra. This information could be stored in the control element. If a spectrum analyzer were built into the control element, a spectral analysis of the reference signal could be made at pre-determined intervals. On the basis of the results of the spectral analysis and the input gains versus reference signal spectrum previously stored in the control element, the gains could be adjusted as the power spectrum of the reference signal changes. This feature would be worthwhile in some cases and not in others. Individual requirements would determine its usefulness. It would require a more complex control element.

The example problems and their solutions indicate that the method which has been developed can be advantageously applied to a large class of tracking problems.

BIBLIOGRAPHY

1. Oldenbourg, R. C., and H. Sartoris, The Dynamics of Automatic Controls, American Society of Mechanical Engineers, New York, 1948.

2. Barker, R. H., "The Pulse Transfer Function and Its Application to Sampling Servo Systems," Proceedings of the Institute of Electrical Engineers, vol. 99, pt. 4, pp. 302-317, December, 1952.

3. Truxal, J. G., Control System Synthesis, McGraw Hill Book Co., Inc., New York, 1955.

4. Tou, J. T., Digital and Sampled-Data Control Systems, McGraw Hill Book Co., Inc., New York, 1959.

5. Weiner, N., Extrapolation, Interpolation, and Smoothing of Stationary Time Series, John Wiley, New York, 1949.

6. Bellman, R., Dynamic Programming, Princeton University Press, Princeton, N. J., 1957.

7. Pischen, J., Disciplines and Techniques of Systems Control, Blaisdell Publishing Co., New York, 1965.

8. Kalman, R. E., and J. E. Bertram, "A Unified Approach to the Theory of Sampling Systems," Journal of the Franklin Institute, vol. 267, p. 408, 1959.

9. Tou, J. T., Optimum Design of Digital Control Systems, Academic Press, Inc., New York, 1963.

10. Kalman, R. E., and R. W. Koephe, "Optimal Synthesis of Linear Sampling Control Systems Using Generalized Performance Indices," Transactions of the American Society of Mechanical Engineers, vol. 80, pp. 1800-1826, November, 1958.

11. Kalman, R. E., "A New Approach to Linear Filtering and Prediction Problems," Journal of Basic Engineering, vol. 82, pp. 35-45, March 1960.

12. Webb, R. P., Synthesis of Measurement Systems, Ph. D. Thesis, Georgia Institute of Technology, July, 1963.

13. Tou, J. T., Modern Control Theory, McGraw Hill Book Co., Inc., New York, 1964.

14. Beckenbach, E. F., Modern Mathematics for the Engineer, McGraw-Hill Book Co., Inc., New York, 1956.

15. Hildebrand, F. B., Methods of Applied Mathematics, Prentice-Hall, Inc., Englewood Cliffs, N. J., 1952.

16. Pipes, L. A., Applied Mathematics for Engineers and Physicists, McGraw-Hill Book Co., New York, 1946.

17. Paryen, E., Modern Probability Theory and Its Applications, John Wiley, New York, 1960.

18. Brunk, H. D., Mathematical Statistics, Ginn and Company, Boston, 1960.

VITA

Jon Willard Petway, the son of Willard J. Petway and Alice Petway, was born in Nashville, Tennessee, on March 10, 1939. He married Judith Margaret Schneidman of Paducah, Kentucky on February 8, 1958, and now has two sons: Will Petway, born February 27, 1962, and Bart Petway, born March 23, 1964.

He was graduated from Paducah Tilghman High School in Paducah, Kentucky, in June, 1957. After receiving the B.S.E.E. degree from the University of Kentucky in 1962, he came to the Georgia Institute of Technology. There he served as a graduate teaching assistant in the School of Electrical Engineering from March, 1962 to September, 1963, and then as an Instructor through September, 1966. He received the M.S.E.E. from Georgia Tech in 1964 and the Ph. D. degree in 1967.